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Équilibre général avec une double infinité d'agents et de biens

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Introduction

En 1874 Léon Walras expliquait dans son livre *Éléments d'Économie Politique Pure*, que les quantités de biens choisies par les acteurs d'une économie (consommateurs et producteurs) ainsi que les prix observés sur le marché pouvaient être interprétés comme une situation d'équilibre. C'était les débuts de la théorie de l'équilibre général.

Entre 1933 et 1936 Abraham Wald donna les premiers résultats rigoureux sur l'existence de solutions au problème posé par Walras. Il souligna en particulier que les théorèmes d'existence ne pourront s'obtenir que par des arguments d'analyse mathématique complexes. Deux décennies plus tard, au début des années 50, de nombreux résultats d'existence d'équilibres furent obtenus indépendamment par McKenzie [38], Arrow et Debreu [8], Gale [23] et Nikaido [39]. L'un des résultats les plus généraux étant celui de Gérard Debreu [18]. Ces théorèmes d'existence marquent un virage important dans l'histoire de la théorie de l'équilibre général. En effet, les techniques de calcul différentiel sur les fonctions d'utilité sont remplacées par les techniques d'analyse fonctionnelle. En particulier l'outil le plus utilisé est le théorème de point fixe de Brouwer, ou sa généralisation par Kakutani. Ces résultats d'existence sont donnés pour des modèles d'économies avec un nombre fini de consommateurs (ou agents) et un nombre fini de biens. Un bien est entendu comme un contrat assurant la remise d'un bien physique ou d'un service, à une date t, à un endroit l, et en fonction de la réalisation de certains événements à la date t. Ce type de modèle économique, présenté dans la monographie classique *Théorie de la Valeur* de Gérard Debreu, est appelé modèle de Arrow-Debreu-McKenzie. On trouve dans la littérature de nombreuses extensions et généralisations de ces théorèmes d'existence d'équilibres.

Dès 1966, Aumann [9] propose un modèle d'économie avec une infinité d'agents. L'espace des agents est dans un premier modélisé par le continuum $[0, 1]$, puis Hildenbrand généralise ce modèle aux espaces mesurés. Cette modélisation de l'ensemble des agents est une formulation mathématique rigoureuse du concept économique selon lequel chaque agent individuel n'a qu'une influence négligeable sur l'activité globale de l'économie. Les économies avec un continuum d'agents peuvent être interprétées comme *l'état limite* d'une économie où le nombre d'agents est très important. En d'autres termes, comme dans d'autres modèles physiques analogues, les propriétés des économies avec un continuum d'agents nous donnent des informations sur le comportement des économies avec un grand nombre d'agents. On trouve des résultats dans ce sens dans Hildenbrand [25] et Kannai [30].

Aumann démontre dans [9] l'existence d'un équilibre pour des économies d'échanges avec des préférences transitives et complètes. Schmeidler [54] généralise ce résultat aux économies avec des préférences incomplètes et Hildenbrand [26, 27] généralise le résultat de Aumann aux économies de production mais toujours avec des préférences transitives et complètes.

Dans le cadre des économies avec un nombre fini d'agents, une des généralisation les plus importantes des hypothèses de Debreu concerne la transitivité et la complétude des préférences. L'intransitivité et l'incomplétude apparaissent, par exemple, lorsque les préférences sont cycliques (voir Sonnenschein [58]) ou définies à partir de plusieurs alternatives non comparables. Mas-Colell dans [34] et Gale et Mas-Colell dans [24] démontrent que les hypothèses de transitivité et complétude des préférences sont superflues dans le modèle de Arrow-Debreu-McKenzie. On peut trouver d'autres généralisations de ces résultats dans Shafer et Sonnenschein [56, 57].

Si on veut prendre en compte les économies à horizon infini, les économies pour lesquelles l'ensemble des caractéristiques (le temps, l'espace ou les qualités physiques) des biens est infini, ou encore les économies dans l'incertain avec une infinité d'états de la nature, on est alors amené à considérer des modèles avec une infinité de biens. Debreu [17] est l'un des premiers à modéliser l'espace des biens par un espace vectoriel topologique et un système de prix par une forme linéaire continue sur l'espace des biens. Dans son modèle, Debreu a besoin d'une hypothèse d'intériorité sur l'ensemble de production. Cette hypothèse est satisfaite dès lors qu'il y a libre disposition sur la production et que le cône positif associé ait un point intérieur. Un exemple d'espace vérifiant cette hypothèse est L_∞, l'ensemble des fonctions réelles mesurables essentiellement bornées, muni de la norme supérieure. Toutefois Radner souligne dans [48], qu'une forme linéaire continue pour la norme de L_∞ est un concept trop général pour modéliser un système de prix. Par contre un système de prix dans L_1, l'ensemble des fonctions réelles intégrables, est un concept plus approprié à une interprétation économique. Bewley dans [13] est le premier à présenter un résultat d'existence d'équilibres pour des économies avec L_∞ comme espace des biens et L_1 comme espace des prix. Cependant ce résultat ne peut pas être directement généralisé à un espace vectoriel ordonné dont le cône positif ne possède pas de point intérieur. En 1983 Aliprantis et Brown [1] furent les premiers à souligner que le cadre approprié pour l'étude de l'équilibre général est la structure d'espace de Riesz. Mas-Colell [36] obtint en 1986 un résultat général d'existence d'équilibres en remplaçant l'hypothèse d'intériorité par une condition dite *d'uniforme propreté*. On trouvera d'autres avancées importantes pour des économies avec un nombre fini d'agents, par exemple dans [2, 3, 4, 5, 7, 10, 19, 21, 22, 33, 37, 44, 49, 50, 61, 59, 60].

Très rapidement, dès le milieu des années 70, des résultats d'existence d'équilibres ont été obtenu pour des économies avec une double infinité d'agents et de biens. Notons que pour traiter la double infinité, on rencontre dans la littérature une autre modélisation de l'infinité d'agents. Au lieu de considérer une économie comme une application définie sur un espace mesuré à valeurs dans l'ensemble des caractéristiques des agents, certains auteurs (comme Mas-Colell [35], Jones [28] ou Podczeck [47]) proposent de modéliser l'infinité d'agents par une distribution sur l'ensemble des caractéristiques.

La littérature la plus importante concerne les économies avec des biens différenciés, par exemple Jones [28], Mas-Colell [35], Ostroy-Zame [43] et Podczeck [45, 47]. Dans Khan et Yannelis [32], Rustichini et Yannelis [52] et Podczeck [45], des résultats d'existence sont obtenus pour des économies avec un espace des biens modélisé par un Banach séparable ordonné dont le cône positif est d'intérieur non vide. On trouvera dans Bewley [14], Podczeck [45] et Zame [65], des résultats d'existence d'équilibres pour des économies avec un espace des biens modélisé par L_∞ et des prix dans L_1.

Dans tous les résultats d'existence d'équilibres avec une infinité d'agents (et un espace des biens de dimension finie ou infinie) cités ci-dessus, les préférences sont transitives et complètes (sauf dans [54] où elles ne sont pas supposées complètes). Khan et Vohra [31] pour un nombre fini de biens et Noguchi [41, 40] pour une infinité de biens, ont tentés de généraliser ces résultats aux économies avec des préférences non ordonnées, c'est à dire intransitives et incomplètes. Les préférences modélisées dans [31, 41, 40] ne sont certes ni transitives ni complètes, mais elles dépendent des consommations des autres agents. Plus précisément, si x est la fonction qui à chaque agent a associe son plan de consommation $x(a)$, alors les préférences d'un agent ne dépendent pas de son panier de consommation individuel $x(a)$ mais de la fonction x. En particulier, comme la fonction x n'a de sens que presque partout, les préférences de l'agent a ne dépendent pas de sa propre consommation $x(a)$. En 2000, Balder [11] a démontré que les hypothèses de mesurabilité et de continuité associées à ce type de modélisation n'étaient pas compatibles, rendant ainsi les théorèmes d'existence vides.

Dans le cadre d'une économie avec un espace mesuré d'agents, nous proposons dans cette thèse de généraliser les résultats d'existence d'équilibres à une économie avec des préférences non ordonnées (intransitives et incomplètes) mais sans externalités. Nous traiterons aussi bien le cas des espaces de biens de dimension finie (chapitre 2) que ceux de dimension infinie (chapitres 3 et 4).

Nous proposons une nouvelle approche pour démontrer l'existence d'un équilibre pour une économie avec un espace mesuré d'agents. On se place dans le cadre où l'économie est définie comme une fonction de l'espace des agents à valeurs dans l'ensemble des caractéristiques. On ne traite pas le modèle d'une économie définie par une distribution sur l'espace des caractéristiques. Notre méthode de preuve consiste à approcher notre économie par une suite d'économies avec un nombre fini, mais de plus en grand, d'agents. Pour chaque économie (avec un nombre fini d'agents) de la suite, on démontre l'existence d'un équilibre en appliquant les nombreux et récents résultats de la littérature (espace des biens de dimension finie, de dimension infinie mais avec condition d'intériorité et de dimension infinie avec propreté uniforme). On démontre alors que la suite d'équilibres ainsi construite converge vers un équilibre de l'économie initiale.

Le premier chapitre est consacré à l'outil mathématique que nous avons développé pour permettre d'approcher une économie *mesurable* par une suite d'économies *simples*. Une fonction réelle f définie sur un espace mesuré $(A, \mathcal{A}, \mu)$ est dite mesurable si elle peut s'écrire comme limite ponctuelle d'une suite de fonctions simples. En particulier, si f est mesurable, il existe une suite de partitions $(\pi^n)_{n\in\mathbb{N}}$

de A approchant la fonction f dans le sens où on peut construire pour chaque $n \in \mathbb{N}$, une fonction f^n constante sur les éléments de la partition π^n et telle que la suite $(f^n)_{n\in\mathbb{N}}$ converge ponctuellement vers f. On démontre dans le chapitre 1 que ce résultat se généralise à une famille dénombrable de correspondances. On démontre ensuite dans les chapitres 2, 3 et 4 que les hypothèses classiques de mesurabilité d'une économie assurent que cette économie est *approchable* par une suite d'économies avec un nombre fini d'agents.

Dans le chapitre 2, on applique les résultats du chapitre 1 aux économies avec un nombre fini de biens. On démontre d'une part un théorème d'existence pour des économies avec des préférences non ordonnées mais convexes, et d'autre part, pour des économies avec des préférences ordonnées mais non convexes. On généralise alors les résultats d'existence d'Aumann [9], Schmeidler [54] et Hildenbrand [26]. En particulier, on démontre que les hypothèses de complétude des préférences et d'irréversibilité de la production sont superflues. On montre aussi que pour l'existence de quasi-équilibres, on peut se passer de l'hypothèse de convexité des ensembles de consommation et que les hypothèses de continuité des préférences peuvent être remplacées par des hypothèses de semi-continuité inférieure.

Dans le chapitre 3, on applique les résultats du chapitre 1 aux économies dont l'espace des biens est modélisé par un Banach séparable[1] ordonné dont le cône positif est d'intérieur non vide. On généralise les résultats d'existence de Podczeck [45], Khan et Yannelis [32] et Rustichini et Yannelis [52], aux économies avec des préférences non ordonnées et un secteur productif non trivial.

Dans le dernier chapitre, on applique les résultats du chapitre 1 aux économies avec des biens différenciés. On se restreint au cadre des biens parfaitement divisibles et on généralise les résultats d'existence de Ostroy et Zame [43] et Podczeck [45], aux économies avec des préférences non ordonnées et un secteur productif non trivial. De plus on remplace les hypothèses classiques sur les taux marginaux de substitution par une hypothèse plus faible d'uniforme propreté, mettant ainsi à profit la structure d'espace de Riesz de l'espace des biens.

Économies avec une double infinité d'agents et de biens

Le modèle

On considère une dualité $\langle \mathbb{P}, \mathbb{L}\rangle$ où $\mathbb{P}$ et $\mathbb{L}$ sont deux espaces vectoriels mis en dualité[2] par $\langle .,. \rangle : \mathbb{P}\times\mathbb{L} \to \mathbb{R}$. On considère un espace mesuré fini complet[3] $(A, \mathcal{A}, \mu)$ et un ensemble fini J. Pour chaque $j \in J$, on considère une fonction positive intégrable θ_j de A dans $\mathbb{R}_+$, vérifiant $\int_A \theta_j(a)d\mu(a) = 1$, et un ensemble $Y_j \subset \mathbb{L}$. De plus, on se donne une fonction sommable[4] e de A dans $\mathbb{L}$, une correspondance X de A dans $\mathbb{L}$ et des préférences P de X, c'est à dire, P est une correspondance de A dans $\mathbb{L} \times \mathbb{L}$ telle que pour tout $a \in A$, $P(a) \subset X(a) \times X(a)$ et $P(a)$ est une relation binaire irréflexive[5] sur $X(a)$.

Une économie $\mathcal{E}$ est une famille de la forme

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{P}, \mathbb{L}\rangle, (X, P, e), (Y_j, \theta_j)_{j\in J}).$$

L'espace des biens de $\mathcal{E}$ est représenté par $\mathbb{L}$ et l'espace des prix est représenté par $\mathbb{P}$. La valeur du panier de biens $x \in \mathbb{L}$ selon le système de prix $p \in \mathbb{P}$ est représentée par $\langle p, x\rangle$.

L'ensemble des agents ou consommateurs est modélisé par A, la tribu $\mathcal{A}$ représente l'ensemble des coalitions admissibles et le réel positif $\mu(E)$ représente la fraction d'agents qui sont dans la coalition $E \in \mathcal{A}$.

[1] Tourky et Yannelis [62], ont démontré que les résultats d'existence de [32] et [52] ne peuvent pas être étendus aux espaces de biens non séparables.

[2] L'application $\langle .,. \rangle$ est bilinéaire et non dégénérée, c'est à dire, étant donné $x \in \mathbb{L}$, si pour tout $p \in \mathbb{P}$, $\langle p, x\rangle = 0$ alors $x = 0$ et symétriquement, étant donné $p \in \mathbb{P}$, si pour tout $x \in \mathbb{L}$, $\langle p, x\rangle = 0$ alors $p = 0$.

[3] L'espace mesuré $(A, \mathcal{A}, \mu)$ est complet si $\mathcal{A}$ contient toutes les parties μ-négligeables de $(A, \mathcal{A}, \mu)$. On rappelle que $E \subset A$ est μ-négligeable s'il existe $B \in \mathcal{A}$ tel que $E \subset B$ et $\mu(B) = 0$.

[4] Une fonction $x : A \to \mathbb{L}$ est dite (scalairement) mesurable lorsque pour tout $p \in \mathbb{P}$, la fonction réelle $\langle p, x(.)\rangle : a \mapsto \langle p, x(a)\rangle$ est mesurable. Une fonction mesurable x de A dans $\mathbb{L}$ est dite intégrable lorsque pour tout $p \in \mathbb{P}$, la fonction réelle $\langle p, x(.)\rangle$ est intégrable. La fonction intégrable x de A dans $\mathbb{L}$ est dite sommable si il existe $v \in \mathbb{L}$ tel que pour tout $p \in \mathbb{P}$, $\langle p, v\rangle = \int_A \langle p, x(a)\rangle\, d\mu(a)$. Alors le vecteur v (unique) est noté $\int_A x(a)d\mu(a)$.

[5] C'est à dire pour tout $x \in X(a)$, $(x, x) \notin P_a(x)$.

Pour chaque agent $a \in A$, l'ensemble de consommation est représenté par $X(a) \subset \mathbb{L}$ et la relation de préférence est représentée par $P(a) \subset X(a) \times X(a)$. On définit la correspondance[6] $P_a : X(a) \twoheadrightarrow X(a)$ par $P_a(x) = \{x' \in X(a) \mid (x, x') \in P(a)\}$, pour tout $x \in X(a)$. En particulier si $x \in X(a)$ est un panier de biens alors $P_a(x)$ est l'ensemble des paniers de biens strictement préférés à x par l'agent a. L'ensemble des plans (ou allocations) de consommations de l'économie $\mathcal{E}$ est l'ensemble $\mathrm{S}^1(X)$ des sélections[7] sommables de X. L'ensemble de consommation agrégé, noté X_Σ, est défini par

$$X_\Sigma := \int_A X(a)d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in \mathrm{S}^1(X) \quad v = \int_A x(a)d\mu(a) \right\}.$$

La dotation initiale de l'agent $a \in A$ est représentée par le panier de biens $e(a) \in \mathbb{L}$. La dotation initiale agrégée est noté $\omega := \int_A e(a)d\mu(a)$.

Le secteur productif de l'économie $\mathcal{E}$ est représenté par un ensemble fini J de producteurs (entreprises) avec des ensembles de production $(Y_j)_{j\in J}$, où pour chaque producteur $j \in J$, $Y_j \subset \mathbb{L}$. Les profits de l'entreprise $j \in J$ sont distribués aux agents selon la fonction de répartition θ_j. L'ensemble des allocations (ou plans) de production est $\mathrm{S}^1(Y) := \prod_{j\in J} Y_j$. L'ensemble de production agrégé est $Y_\Sigma := \sum_{j\in J} Y_j$.

Exemples

Le choix le plus naturel pour l'espace des biens est $\mathbb{L} = \mathbb{R}^\ell$ où ℓ est le nombre de biens physiques présents dans le marché. Pour un panier de biens $x = (x_1, \cdots, x_\ell) \in \mathbb{R}^\ell$, x_i représente la quantité d'unités du bien i, présente dans le panier. Dans le chapitre 2 on démontre l'existence d'un équilibre de Walras pour des économies avec un espace des biens de dimension finie.

Il existe des situations économiques pour lesquelles le contexte de la dimension finie n'est pas satisfaisant pour modéliser l'espace des biens. Prenons l'exemple d'une économie avec un seul bien physique mais dont l'ensemble T des caractéristiques le définissant est infini. Deux situations sont modélisables. Dans la première, un panier de biens est défini par une fonction x continue de T dans $\mathbb{R}$. Alors pour chaque caractéristique $t \in T$, $x(t)$ représente la quantité d'unités du bien t, présente dans le panier. Dans ce modèle, une liste des prix p est modélisée par une mesure borélienne finie sur T où pour tout borélien B de T, $p(B)$ représente le prix moyen des biens dont les caractéristiques sont dans B. Ainsi la valeur du panier x selon la liste des prix p est

$$\int_T x(t)dp(t).$$

Naturellement, si T est fini, ce modèle coïncide avec le modèle fini. Cette dualité *prix-biens* $\langle M(T), C(T)\rangle$ est un cas particulier de la dualité traitée dans le chapitre 3.

La seconde situation modélisable, lorsque l'ensemble des caractéristiques est un espaces métrique compact T, est la suivante. Un panier de biens x est défini par une mesure borélienne finie sur T. En particulier pour chaque borélien B de T, $x(B)$ représente la quantité, présente dans le panier, d'unité de biens dont les caractéristiques sont dans l'ensemble B. Une liste des prix est une fonction continue p de T dans $\mathbb{R}$, où pour chaque t, $p(t)$ représente le prix d'une unité du bien t. La valeur du panier x selon la liste de prix p est

$$\int_T p(t)dx(t).$$

Une nouvelle fois, si T est fini, ce modèle coïncide avec le modèle fini. Cette dualité *prix-biens* $\langle C(T), M(T)\rangle$ est traitée dans le chapitre 4.

Il existe bien d'autres exemples de modèles où l'espace des biens est de dimension infinie. Pour le modèle d'économies à allocations inter-temporelles, on prend en compte une infinité de dates possibles dans la constitution d'un panier de biens. Ainsi l'espace des biens peut être modélisé par ℓ^∞, $\mathcal{L}^\infty([0,T], \mathcal{B}, \lambda)$ ou $\mathcal{L}^\infty([0,+\infty[, \mathcal{B}, \lambda)$ avec $\mathcal{B}$ la tribu des boréliens et λ la mesure de Lebesgue.

[6] Notons que la relation binaire $P(a)$ coïncide avec le graphe de la correspondance P_a.

[7] La fonction $x : A \to \mathbb{L}$ est une sélection de la correspondance X si pour presque tout $a \in A$, $x(a)$ est dans $X(a)$.

Dans ce cas, par souci de signification économique l'espace des prix est modélisé par ℓ^1, $\mathcal{L}^1([0,T],\mathcal{B},\lambda)$ ou $\mathcal{L}^1([0,+\infty[,\mathcal{B},\lambda)$. Pour le modèle d'économies à allocations dans l'incertain, la consommation dépend de l'état du monde (avec une infinité d'états possibles), représenté par un espace probabilisé (Ω,Σ,μ). L'espace des biens et l'espace de prix sont alors modélisés par $\mathcal{L}^2(\Omega,\Sigma,\mu)$.

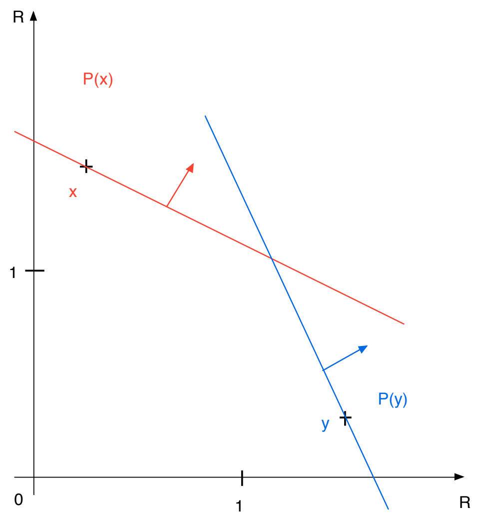
Dans la littérature, l'ensemble des d'agents est souvent modélisé par le segment unité $[0,1]$ muni de la tribu de Lebesgue et de la mesure de Lebesgue. Dans ce modèle, chaque agent a le même "poids". On peut aussi modéliser l'ensemble des types d'agents par le segment $[0,100]$ où chaque $t\in[0,100]$, représente un âge. Alors la mesure μ choisie est la mesure induite par la "pyramide des âges". C'est à dire, si B est une partie Lebesgue mesurable de $[0,100]$, alors $\mu(B)$ représente la fraction de la population dont l'âge est dans la partie B.

Préférences non ordonnées

Un des apports significatif de cette thèse est de généraliser certains résultats d'existence d'équilibres aux économies avec des préférences non ordonnées ou partiellement ordonnées. Avant de présenter les résultats démontrés dans les chapitres 2, 3 et 4, nous rappelons la définition de préférences ordonnées et partiellement ordonnées. Soit X un ensemble et $P\subset X\times X$ une relation binaire sur X. La relation P est dite partiellement ordonnée si elle est irréflexive ($(x,x)\notin P$, pour tout $x\in X$) et transitive ($[(x,y)\in P$ et $(y,z)\in P]$ implique $(x,z)\in P$, ceci pour tout $(x,y,z)\in X^3$). La relation P est dite ordonnée si elle est irréflexive, transitive et négativement transitive ($[(x,y)\notin P$ et $(y,z)\notin P]$ implique $(x,z)\notin P$, ceci pour tout $(x,y,z)\in X^3$). Notons que lorsque P est ordonnée, alors la relation binaire R sur X définie par $R:=\{(x,y)\in X^2 \mid (y,x)\notin P\}$, est réflexive ($(x,x)\in P$, pour tout $x\in X$), transitive et complète (pour tout $(x,y)\in X^2$, on a $[(x,y)\in R$ ou $(y,x)\in R]$). Dans la littérature, lorsque P est ordonnée, elle est souvent notée $\succ$ et la relation R associée est notée $\preceq$.

Exemple. On se propose de donner dans $\mathbb{R}^2$, un exemple de relation de préférence non transitive. On prend $X=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\geqslant 0 \text{ et } x_2\geqslant 0\}$. Pour chaque $u\geqslant 0$, on note $p(u):=(1,u)\in\mathbb{R}^2$. On définit maintenant la relation de préférence P comme suit :

$$\forall x:=(x_1,x_2)\in X \quad P(x):=\{x'\in X \mid \langle p(x_2),x'-x\rangle>0\}.$$



La relation P est continue, i.e., pour tout $x\in X$, $P(x)$ et $P^{-1}(x)=\{x'\in X \mid x\in P(x')\}$ sont ouverts dans X, mais elle n'est pas transitive. De plus il n'existe pas de relation $\tilde{P}$ ordonnée et continue dominant P, c'est à dire vérifiant pour tout $x\in X$, $P(x)\subset\tilde{P}(x)$.

Le concept d'équilibre de Walras

Définition 1. Un *équilibre de Walras* de l'économie $\mathcal{E}$ est un triplet (x^*, y^*, p^*) de $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{P}$ avec $p^* \neq 0$ et vérifiant les propriétés suivantes.

(a) Pour presque tout $a \in A$,

$$\langle p^*, x^*(a) \rangle = \langle p^*, e(a) \rangle + \sum_{j \in J} \theta_j(a) \left\langle p^*, y_j^* \right\rangle$$

et

$$x \in P_a(x^*(a)) \Longrightarrow \langle p^*, x \rangle > \langle p^*, x^*(a) \rangle .$$

(b) Pour tout $j \in J$,

$$y \in Y_j \Longrightarrow \langle p^*, y \rangle \leqslant \left\langle p^*, y_j^* \right\rangle .$$

(c)

$$\int_A x^*(a) d\mu(a) = \int_A e(a) d\mu(a) + \sum_{j \in J} y_j^* .$$

Un *quasi-équilibre de Walras* d'une économie $\mathcal{E}$ est un triplet (x^*, y^*, p^*) de $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{P}$ avec $p^* \neq 0$ et vérifiant les conditions (b), (c) et (a') définie par

(a') pour presque tout $a \in A$,

$$\langle p^*, x^*(a) \rangle = \langle p^*, e(a) \rangle + \sum_{j \in J} \theta_j(a) \left\langle p^*, y_j^* \right\rangle$$

et

$$x \in P_a(x^*(a)) \Longrightarrow \langle p^*, x \rangle \geqslant \langle p^*, x^*(a) \rangle .$$

Un équilibre de Walras est évidement un quasi-équilibre de Walras. Nous proposons dans la remarque suivante, une condition suffisante pour que la réciproque soit vrai.

Remarque. Soit (x^*, y^*, p^*) un quasi-équilibre de Walras d'un économie $\mathcal{E}$. Si pour presque tout agent $a \in A$, il existe $x^0(a) \in X(a)$ et $y^0(a) \in \sum_{j \in J} \theta_j(a) Y_j$ tel que

$$\left\langle p^*, x^0(a) \right\rangle < \langle p^*, e(a) \rangle + \left\langle p^*, y^0(a) \right\rangle ,$$

et tel que $X(a)$ est étoilé[8] en $x^0(a)$ et l'ensemble des préférés $P_a(x^*(a))$ est radial[9] vers $x^0(a)$, alors (x^*, y^*, p^*) est un équilibre de Walras.

Les hypothèses générales

Dans chacun des chapitres 2, 3 et 4, nous donnons une liste d'hypothèses suffisantes pour qu'une économie $\mathcal{E}$ possède un équilibre de Walras. Certaines hypothèses que doit vérifier l'économie $\mathcal{E}$ vont dépendre de la dualité *prix-biens* $\langle \mathbb{P}, \mathbb{L} \rangle$, par contre d'autres sont indépendantes du choix de la dualité. Nous présentons ici la liste des hypothèses communes aux trois applications traitées dans les chapitres 2, 3 et 4.

Nous supposons donné $\mathbb{L}_+ \subset \mathbb{L}$ un cône convexe (de sommet 0) fermé saillant[10] définissant[11] un ordre partiel sur $\mathbb{L}$, noté $\geqslant$. Ce cône $\mathbb{L}_+$ sera appelé le cône positif.

[8]Un sous ensemble $X \subset \mathbb{L}$ est étoilé en x^0 si pour tout $x \in X$, le segment $[x^0, x]$ reste dans X.

[9]Une partie P de X est radiale vers x^0 si pour tout $x \in P$, il existe $\lambda > 0$ tel que le segment $[x, x^0 + \lambda(x - x^0)]$ reste dans P.

[10]C'est à dire $\mathbb{L}_+ \cap (-\mathbb{L}_+) = \{0\}$. Ainsi un cône convexe saillant ne contient pas de droite.

[11]Pour tout $(x, y) \in \mathbb{L}$, $x \geqslant y$ lorsque $x - y \in \mathbb{L}_+$.

Sur $\mathbb{L}$ on considère la topologie faible $\sigma(\mathbb{L},\mathbb{P})$ et la topologie de Mackey $\tau(\mathbb{L},\mathbb{P})$. Rappelons que l'ensemble des parties $\sigma(\mathbb{L},\mathbb{P})$-fermées convexes et l'ensemble des parties $\tau(\mathbb{L},\mathbb{P})$-fermées convexes coïncident. Ainsi, on dira qu'un sous ensemble X est fermé convexe pour $\sigma(\mathbb{L},\mathbb{P})$-fermé convexe ou $\tau(\mathbb{L},\mathbb{P})$-fermé convexe. En particulier l'enveloppe convexe fermée d'une partie $X \subset \mathbb{L}$ sera notée $\overline{\mathrm{co}}\, X$ et l'enveloppe convexe de X sera notée $\mathrm{co}\, X$. De même les parties $\sigma(\mathbb{L},\mathbb{P})$-bornées[12] et $\tau(\mathbb{L},\mathbb{P})$-bornées de $\mathbb{P}$ coïncident. Dans la suite, si τ est une topologie sur $\mathbb{L}$, l'intérieur pour τ d'une partie $X \subset \mathbb{L}$ sera notée $\tau - \mathrm{int}\, X$.

Hypothèse (C). ***[Consommateurs]*** *Pour presque tout agent $a \in A$,*

(a) *l'ensemble de consommation $X(a)$ est convexe fermé,*

(b) *pour chaque panier $x \in X(a)$, $P_a(x)$ est $\tau(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$ et $P_a^{-1}(x)$[13] est $\sigma(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$,*

(c) *la relation de préférence $P(a)$ est convexe, i.e., pour chaque panier $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$, et lorsque $\mathbb{L}$ est de dimension infinie, si a est dans la partie non-atomique[14] de $(A,\mathcal{A},\mu)$, alors $X(a) \setminus P_a^{-1}(x)$ est convexe.*

Remarque. Lorsque $X(a) \setminus P_a^{-1}(x)$ est supposé convexe, l'ensemble $P_a^{-1}(x)$ est $\sigma(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$ si et seulement si il est $\tau(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$. Dans la littérature, la condition $x \notin \mathrm{co}\, P_a(x)$ est souvent remplacée par $P_a(x)$ est convexe. Dans ce cas $P_a(x)$ est $\tau(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$ si et seulement si $P_a(x)$ est $\sigma(\mathbb{L},\mathbb{P})$-ouvert dans $X(a)$.

Remarque. Lorsque $P(a)$ est partiellement ordonnée, supposer que pour tout $x \in X(a)$, $X(a) \setminus P_a^{-1}(x)$[15] est convexe, implique que pour tout $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$. En particulier l'hypothèse C est automatiquement vérifiée sous les hypothèses (A1-4) dans Podczeck [47], les hypothèses (3.1) et (3.2) dans Khan et Yannelis [32] et sous les hypothèses P1-4 pour les marchés "*economically thick*" d' Ostroy et Zame [43].

Hypothèse (M). ***[Mesurabilité]*** *Le graphe de la correspondance X est mesurable, c'est à dire*

$$\{(a,x) \in A \times \mathbb{L} \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L})$$

et le graphe de la correspondance des préférences est mesurable, c'est à dire

$$\{(a,x,y) \in A \times \mathbb{L} \times \mathbb{L} \mid (x,y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}) \otimes \mathcal{B}(\mathbb{L}).$$

Remarque. Sous les conditions de l'hypothèse C, si les préférences sont ordonnées, on peut (voir la proposition 1.4.5) remplacer la mesurabilité du graphe de P par la Aumann mesurabilité, c'est à dire, pour toutes sélections mesurables[16] x et y de X,

$$\{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

Remarque. En appliquant la proposition 1.4.5, l'hypothèse M est alors vérifiée dans les modèles de Aumann [9], Schmeidler [54], Hildenbrand [26], Khan et Yannelis [32], Podczeck [45] et Ostroy et Zame [43].

Hypothèse (P). ***[Producteurs]*** *L'ensemble de production agrégé Y_Σ est non vide, convexe fermé et satisfait $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$.*

Remarque. Cette hypothèse est commune à toute la littérature traitant des économies de production avec un espace mesuré d'agents.

[12] On rappelle que dans un espace topologique $(\mathbb{L},\tau)$, une partie $B \subset \mathbb{L}$ est τ-bornée, si pour tout τ-voisinage V de 0, il existe $t > 0$ tel que $B \subset tV$.

[13] Pour chaque $y \in X(a)$, $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$.

[14] Un élément $E \in \mathcal{A}$ est un atome de $(A,\mathcal{A},\mu)$ si $\mu(E) \neq 0$ et $[B \in \mathcal{A}$ et$B \subset E]$ implique $\mu(B) = 0$ ou $\mu(E \setminus B) = 0$.

[15] Si $P(a)$ est ordonnée alors $X(a) \setminus P_a^{-1}(x) = \{y \in X(a) \mid y \succeq_a x\}$.

[16] Une fonction mesurable $x : A \to \mathbb{L}$ est une sélection mesurable d'une correspondance $X : A \twoheadrightarrow \mathbb{L}$ si pour presque tout $a \in A$, $x(a) \in X(a)$.

Hypothèse (S). *[**Survie**] Pour presque tout $a \in A$,*[17]

$$0 \in \left(\{e(a)\} + \sum_{j \in J} \theta_j(a) \overline{\mathrm{co}}\, Y_j + A(Y_\Sigma) - X(a) \right).$$

Remarque. L'hypothèse S traduit le besoin de compatibilité entre les ressources et les consommations possibles. Dans la littérature, il est souvent supposé que pour presque tout agent $a \in A$, $e(a) \in X(a)$ et pour tout producteur $j \in J$, $0 \in Y_j$.

Hypothèse (BI). *[**Borné Inférieurement**] La correspondance X est intégralement bornée inférieurement*[18]*, la fonction de dotations initiales e est intégralement bornée et l'ensemble agrégé de production libre $Y_\Sigma \cap \mathbb{L}_+$ est borné.*

Remarque. Notons que si il existe sur $\mathbb{L}$ une norme $\|.\|$ telle que $(\mathbb{L}, \|.\|)' = \mathbb{P}$ (c'est le cas des modèles traités dans les chapitres 2 et 3), alors toute partie de $\mathbb{L}$ est bornée (pour la dualité $\langle \mathbb{P}, \mathbb{L} \rangle$) si et seulement si elle est bornée pour la norme $\|.\|$. De même, si il existe sur $\mathbb{P}$ une norme $\|.\|$ telle que $(\mathbb{P}, \|.\|)' = \mathbb{L}$ (c'est le cas du modèle traité dans le chapitre 4), alors toute partie de $\mathbb{L}$ est bornée (pour la dualité $\langle \mathbb{P}, \mathbb{L} \rangle$) si et seulement si elle est bornée pour la norme duale $\|.\|^*$.

Remarque. L'hypothèse de bornitude sur l'ensemble de production libre est plus faible que l'hypothèse faite dans Hildenbrand [26] et Podczeck [46], où l'ensemble agrégé de production libre est supposé trivial, c'est à dire $Y_\Sigma \cap \mathbb{L}_+ = \{0\}$.

Hypothèse (Lns). *[**Non Satiété Locale**] Pour presque tout agent $a \in A$, pour tout $x \in X(a)$,*

(i) *si $P_a(x) = \emptyset$, alors $x \geqslant e(a) + \sum_{j \in J} \theta_j(a) y_j$, pour tout $y \in \prod_{j \in J} Y_j$;*

(ii) *si x n'est pas un panier de satiété, alors $x \in \overline{\mathrm{co}}\, P_a(x)$.*

Remarque. Cette hypothèse est en particulier vérifiée lorsqu'il y a non satiation locale partout, i.e., pour presque tout agent $a \in A$, pour tout $x \in X(a)$, $x \in \overline{\mathrm{co}}\, P_a(x)$. Dans Podczeck [45] et [47], les économies sont des économies de libre échange avec libre disposition, i.e., pour tout $j \in J$, $Y_j = -\mathbb{L}_+$. Ainsi les hypothèses $B4-5$ dans [45] et $C5-6$ dans [47] impliquent l'hypothèse Lns.

Dans la suite, les economies considérées sont supposées satisfaire les **hypothèses générales** : C, M, P, S, BI et Lns.

Existence d'équilibres avec un nombre fini de biens

Nous supposons dans cette partie que l'espace des biens $\mathbb{L}$ est de dimension finie. L'espace des prix $\mathbb{P}$ est alors modélisé par $\mathbb{L}^*$ le dual algébrique de $\mathbb{L}$. La dualité $\langle ., . \rangle$ est la dualité naturelle définie par $\langle p, x \rangle = p(x)$ pour tout $(p, x) \in \mathbb{L}^* \times \mathbb{L}$. Le cône positif $\mathbb{L}_+ \subset \mathbb{L}$ est un cône convexe fermé saillant quelconque. Notons que pour un espace des biens $\mathbb{L}$ de dimension finie, une fonction de A dans $\mathbb{L}$ est sommable dès qu'elle est intégrable.

Remarque. Dans Hildenbrand [26], l'espace des biens est $\mathbb{L} = \mathbb{R}^\ell$, pour un entier $\ell \in \mathbb{N}$, et comme c'est la version de Schmeidler [55] du lemme de Fatou qui est utilisée, le cône positif est $\mathbb{L}_+ = (\mathbb{R}_+)^\ell$. Ici, nous appliquons une version de lemme de Fatou plus récente, démontrée par Cornet et Topuzu [55] (théorème 2.7.2), qui nous permet de considérer des cônes positifs plus généraux. Notons que notre cône positif $\mathbb{L}_+$ n'est pas supposé avoir de points intérieurs.

Les hypothèses générales sont suffisantes pour qu'une économie $\mathcal{E}$ possède un quasi-équilibre de Walras. Pour démontrer qu'un quasi-équilibre de $\mathcal{E}$ est en fait un équilibre, on introduit l'hypothèse suivante.

[17] Le cône asymptotique $A(Z)$ d'une partie convexe $Z \subset \mathbb{L}$ d'un espace vectoriel $\mathbb{L}$ est l'ensemble $\{v \in \mathbb{L} \mid Z + \{v\} \subset Z\}$.

[18] C'est à dire, il existe une fonction sommable $\underline{x} : A \to \mathbb{L}$, intégralement bornée, telle que pour presque tout $a \in A$, $X(a) \subset \{\underline{x}(a)\} + \mathbb{L}_+$. Une fonction $x : A \to \mathbb{L}$ est dite intégralement bornée si il existe une fonction positive réelle intégrable ρ et une partie $V \subset \mathbb{L}$ absolument convexe, bornée et fermée, telle que pour presque tout $a \in A$, $x(a) \in \rho(a)V$.

Hypothèse (SS). *[**Survie Forte**] Pour presque tout agent $a \in A$,*

$$\left(\{e(a)\} + \sum_{j \in J} \theta_j(a) Y_j + A(Y_\Sigma) - X(a)\right) \cap int \mathbb{L}_+ \neq \emptyset.$$

Remarque. On peut remplacer l'hypothèse SS par la condition $(\{\omega\} + Y_\Sigma - X_\Sigma) \cap \operatorname{int} \mathbb{L}_+ \neq \emptyset$ et par une hypothèse d'irréductibilité comme celle utilisée dans Yamazaki [64].

Nous pouvons maintenant énoncer notre principal théorème d'existence pour les économies avec un nombre fini de biens et des préférences non ordonnées.

Théorème 1. *Si l'économie $\mathcal{E}$ vérifie les* **hypothèses générales***, alors il existe un quasi-équilibre de Walras (x^*, y^*, p^*) avec*[19]*$p^* > 0$. Si de plus $\mathcal{E}$ vérifie **SS** alors (x^*, y^*, p^*) est un équilibre de Walras.*

Remarque. Le théorème 1 généralise le théorème 1 de Hildenbrand [26] aux économies avec des préférences non ordonnées. Pour démontrer l'existence d'un quasi-équilibre, nous n'avons pas besoin de supposer que l'ensemble de production agrégé Y_Σ satisfait la propriété d'irréversibilité $Y_\Sigma \cap (-Y_\Sigma) = \{0\}$. De plus nous remplaçons l'hypothèse d'impossibilité de production libre $Y_\Sigma \cap \mathbb{L}_+ = \{0\}$, par l'hypothèse de bornitude de l'ensemble de production libre. Le lemme de Fatou démontré par Cornet et Topuzu [16] nous permet de considérer un cône positif plus général que le cône $(\mathbb{R}_+)^\ell$ quand $\mathbb{L} = \mathbb{R}^\ell$ pour un entier $\ell \in \mathbb{N}$.

Remarque. Nous démontrons dans le chapitre 2, un résultat d'existence plus général. En particulier, nous traitons le cas des préférences partiellement ordonnées (peut être incomplètes) mais non convexes.

Existence d'équilibres avec une infinité de biens

Lorsque l'espace des biens est un espace de Banach séparable $(\mathbb{L}, \|.\|)$, l'espace des prix $\mathbb{P}$ est modélisé par $\mathbb{L}' = (\mathbb{L}, \|.\|)'$, le dual topologique de $\mathbb{L}$. La dualité $\langle ., . \rangle$ est la dualité naturelle définie par $\langle p, x \rangle = p(x)$ pour tout $(p, x) \in \mathbb{L}' \times \mathbb{L}$. La topologie de la norme[20] sur $\mathbb{L}$ sera notée s, la topologie faible $\sigma(\mathbb{L}, \mathbb{L}')$ sera notée w et la topologie faible étoile $\sigma(\mathbb{L}', \mathbb{L})$ sur $\mathbb{L}'$ sera notée w^*. Nous supposons ici que le cône positif $\mathbb{L}_+$ possède un s-point intérieur. Notons qu'une fonction x de A dans $\mathbb{L}$ est (scalairement) mesurable, si et seulement si elle est Bochner[21] mesurable. De plus une fonction mesurable x de A dans $\mathbb{L}$ est intégralement bornée, si et seulement si elle est Bochner[22] intégrable. Notons que si une fonction x de A dans $\mathbb{L}$ est Bochner intégrable, alors elle est sommable.

Nous présentons maintenant les hypothèses suffisantes (en plus des hypothèses générales) pour qu'une économie $\mathcal{E}$ possède un équilibre de Walras.

Hypothèse (B). *[**Borné**] La correspondance X est intégralement bornée*[23]*, à valeurs w-compactes.*

Remarque. En dimension infinie, il existe un lemme de Fatou pour des correspondances bornées (Lemma 6.6 dans Podczeck [45]), par contre, il n'existe pas (encore) de lemme de Fatou pour des correspondances bornées inférieurement. Nous retrouvons donc cette hypothèse dans toute la littérature traitant de ce modèle de double infinité : Khan et Yannelis [32], Podczeck [45], [47] et Rustichini et Yannelis [52]. Notons que sous les hypothèses M, S et B, l'ensemble de consommation agrégé X_Σ est non vide.

[19] C'est à dire $p^* \neq 0$ et pour tout $x \in \mathbb{L}_+$, $p^*(x) \geqslant 0$.

[20] Rappelons que la topologie s coïncide avec la topologie de Mackey $\tau(\mathbb{L}, \mathbb{L}')$.

[21] Une fonction x de A dans $\mathbb{L}$ est Bochner mesurable si il existe une suite $(s_n)_{n \in \mathbb{N}}$ de fonctions simples de A dans $\mathbb{L}$ telle que pour presque tout $a \in A$, $\lim_n \|x(a) - s_n(a)\| = 0$.

[22] Une fonction Bochner mesurable x de A dans $\mathbb{L}$ est Bochner intégrable si il existe une suite $(s_n)_{n \in \mathbb{N}}$ de fonctions simples de A dans $\mathbb{L}$ telle que la fonction réelle $a \mapsto \|x(a) - s_n(a)\|$ est intégrable et $\lim_n \int_A \|x(a) - s_n(a)\| \, d\mu(a) = 0$.

[23] Une correspondance X de A dans $\mathbb{L}$ est intégralement bornée si il existe une fonction positive réelle intégrable ρ et une partie $V \subset \mathbb{L}$ absolument convexe, bornée et fermée, telle que pour presque tout $a \in A$, $X(a) \subset \rho(a)V$. En particulier, comme $\mathbb{L}' = (\mathbb{L}, \|.\|)'$, la correspondance X est intégralement bornée si il existe une fonction positive réelle intégrable ρ telle que pour presque tout $a \in A$, pour tout $x \in X(a)$, $\|x\| \leqslant \rho(a)$.

Pour démontrer qu'un quasi-équilibre de $\mathcal{E}$ est en fait un équilibre, on introduit l'hypothèse suivante.

Hypothèse (SS). *[**Survie Forte**] Pour presque tout agent $a \in A$,*

$$\left(\{e(a)\} + \sum_{j\in J} \theta_j(a)Y_j + A(Y_\Sigma) - X(a)\right) \cap s - int\mathbb{L}_+ \neq \emptyset.$$

Remarque. Pour les économies de libre échange, Podczeck [45], [47] et Khan et Yannelis [32] supposent que pour presque tout agent $a \in A$, $[\{e(a)\} - X(a)] \cap s - \operatorname{int} \mathbb{L}_+ \neq \emptyset$. Cette hypothèse implique notre hypothèse SS.

Nous pouvons maintenant énoncer le théorème d'existence d'un équilibre de Walras pour des économies avec une double infinité d'agents et de biens.

Théorème 2. *Sous les hypothèses générales, si l'économie $\mathcal{E}$ satisfait **B**, alors il existe un quasi-équilibre (x^*, y^*, p^*), avec $p^* > 0$. Si de plus $\mathcal{E}$ satisfait **SS**, alors (x^*, y^*, p^*) est un équilibre de Walras.*

Remarque. Pour les économies avec des préférences convexes, le théorème 2 généralise le théorème 5.1. dans Podczeck [45], aux économies avec des préférences non ordonnées et un secteur productif non trivial. Sous l'hypothèse Lns, le théorème 2 généralise le théorème principal dans Khan et Yannelis [32], aux économies avec des préférences non ordonnées et un secteur productif non trivial.

Remarque. Nous démontrons un théorème un peu plus général dans le chapitre 3. En particulier, nous ne traitons ici que le cas des préférences non ordonnées mais convexes. Le cas des préférences partiellement ordonnées (peut être incomplètes) mais non convexes, est détaillé dans le chapitre 3.

Existence d'équilibres avec des biens différenciés

Nous supposons dans cette partie, que l'espace des biens est $M(T)$, l'ensemble des mesures de Radon sur un espace métrique compact T, et que l'espace des prix est modélisé par $C(T)$, l'ensemble des fonctions réelles continues sur T. La dualité $\langle ., . \rangle$ est la dualité naturelle définie par $\langle p, x\rangle = \int_T p(t)dx(t)$. Chaque point de T représente la description complète de toutes les caractéristiques d'un certain bien physique. Si $x \in M(T)$ est un panier de biens, alors pour chaque borélien $B \subset T$, $x(B)$ indique la quantité totale de biens ayant leurs caractéristiques dans B. Comme chaque élément de $M(T)$ représente un panier de biens potentiel, nous supposons donc comme dans les modèles de Jones [28, 29] et Ostroy et Zame [43] mais différemment de ceux de Mas-Colell [35] et Cornet et Médecin [15], que tous les biens sont parfaitement divisibles. Si $p \in C(T)$, alors pour chaque $t \in T$, $p(t)$ est interprété comme la valeur (ou le prix) d'une unité du bien ayant la caractéristique t. On note w^* la topologie faible étoile $\sigma(M(T), C(T))$ et bw^* la topologie la plus fine sur $M(T)$ qui coïncide avec w^* sur les ensembles w^*-compacts. Les boréliens de $(M(T), w^*)$ et $(M(T), bw^*)$ coïncident et l'ensemble de ces boréliens est noté $\mathcal{B}$.

Notons que dans ce cas particulier de dualité *prix-biens*, une fonction de A dans $M(T)$ (scalairement) mesurable est dite Gelfand mesurable, et une fonction (scalairement) intégrable est dite Gelfand intégrable. Notons que toute fonction Gelfand intégrable de A dans $M(T)$ est automatiquement sommable.

Nous présentons maintenant les hypothèses suffisantes (en plus des hypothèses générales) pour qu'une économie $\mathcal{E}$ possède un équilibre de Walras.

Hypothèse (MON). *[**Monotonie**] Pour presque tout agent $a \in A$, la relation de préférence $P(a)$ est monotone, c'est à dire,*

$$\forall m \in M(T)_+ \quad \exists \alpha > 0 \quad x + \alpha m \in P_a(x) \cup \{x\}.$$

Hypothèse (E). ***[Dotations initiales]*** *Il existe* $\overline{v} \in X_\Sigma$ *et* $\overline{u} \in Y_\Sigma$ *tel que*[24] $\omega + \overline{u} - \overline{v} \gg 0$.

Remarque. On suppose dans l'hypothèse E, que tous les biens sont présents dans le marché. Dans la littérature des biens différenciés, les ensembles de consommations coïncident avec le cône positif $M(T)_+$. Ainsi si on suppose que $\omega \gg 0$ (par exemple dans [28, 29, 43, 45]) ou que $\omega + \overline{u} \gg 0$ (dans [46]), alors l'hypothèse E est vérifiée.

Remarque. En dimension infinie, il existe un lemme de Fatou pour des correspondances bornées (Lemma 6.6 dans Podczeck [45]), par contre, il n'existe pas (encore) de lemme de Fatou pour des correspondances bornées inférieurement. Les hypothèses MON et E vont nous permettre de montrer que chaque prix d'équilibre p^* est strictement positif, c'est à dire que pour tout $t \in T$, $p^*(t) > 0$. Ceci nous permettra de "contrôler" la norme des plans de consommations d'équilibre et d'appliquer le lemme de Fatou pour des correspondances bornées.

Hypothèse (UP). ***[Propreté Uniforme]*** *Il existe un cône* Γ*,* bw^**-ouvert tel que* $\Gamma \cap M(T)_+ \neq \emptyset$ *et tel que pour presque tout* $a \in A$*, pour tout* $j \in J$*, pour chaque* $(x, y) \in X(a) \times Y_j$*,*

(a) *il existe un ensemble* $A^a_x \subset M(T)$*, radial*[25] *en* x*, tel que* [26]

$$(\{x\} + \Gamma) \cap \{z \in M(T) \mid z \geqslant x \wedge e(a)\} \cap A^a_x \subset \overline{\mathrm{co}}\, P_a(x) \ ;$$

(b) *il existe un ensemble* $A^j_y \subset M(T)$*, radial en* y*, tel que*

$$(\{y\} - \Gamma) \cap \{z \in M(T) \mid z \leqslant y \vee 0\} \cap A^j_y \subset \overline{\mathrm{co}}\, Y_j.$$

Remarque. Cette hypothèse est inspirée de l'hypothèse de F-propreté introduite par Podczeck [44] pour des économies d'échanges et adaptée aux économies de production par Florenzano et Marakulin [22]. On pourra trouver une étude plus détaillée des différentes hypothèses de propreté de la littérature dans Aliprantis, Tourky et Yannelis [6].

Remarque. L'hypothèse UP est plus faible que les hypothèses C3 et P4 dans Podczeck [46], puisque les ensembles radiaux A^a_x et A^j_y sont supposés coïncider avec $M(T)$. Ainsi d'après les propositions 3.2.1 et 3.3.1 dans [46], l'hypothèse UP est plus faible que les habituelles hypothèses sur les taux marginaux de substitution dans les modèles avec des biens différenciés, par exemple dans Jones [28, 29], Ostroy et Zame [43] et Podczeck [45].

Pour démontrer qu'un quasi-équilibre de $\mathcal{E}$ est en fait un équilibre, on introduit l'hypothèse suivante.

Hypothèse (S'). *Pour presque tout agent* $a \in A$*,*

$$\left(\{e(a)\} + \sum_{j \in J} \theta_j(a) \overline{\mathrm{co}}\, Y_j - X(a)\right) \cap M(T)_+ \neq \{0\}.$$

Remarque. Sous les hypothèses C et S', chaque quasi-équilibre (x^*, y^*, p^*) avec[27] $p^* \gg 0$ est en fait un équilibre de Walras. Cette hypothèse peut être remplacée par les hypothèses habituelles d'irréductibilité adaptées à notre contexte, voir Podczeck [47].

Théorème 3. *Sous les hypothèses générales, si l'économie* $\mathcal{E}$ *satisfait* ***MON, E*** *et* ***UP****, alors il existe un quasi-équilibre* (x^*, y^*, p^*)*, avec* $p^* \gg 0$*. Si de plus* $\mathcal{E}$ *satisfait* ***S'****, alors* (x^*, y^*, p^*) *est en fait un équilibre de Walras.*

[24] Pour $x \in M(T)$, on note $x \gg 0$ lorsque pour tout $p \in C(T)_+$, $\langle p, x \rangle > 0$. En particulier, si V est un ouvert non vide de T, alors $x(V) > 0$.

[25] Un ensemble $A \subset M(T)$ est radial en $x \in A$ si pour tout $y \in M(T)$, il existe $\lambda > 0$ tel que le segment $[x, x + \lambda y]$ reste dans A.

[26] Si $(x, y) \in M(T)$ alors la borne supérieure de $\{x, y\}$ est noté $x \vee y$ et la borne inférieure est notée $x \wedge y$.

[27] Pour $p \in C(T)$, on note $p^* \gg 0$ lorsque pour tout $t \in T$, $p^*(t) > 0$.

Remarque. Ce théorème généralise aux économies avec des préférences non ordonnées et un secteur productif non trivial, les résultats d'existence de Ostroy et Zame [43] (théorèmes 1.a et 3.a) et ceux (dans le contexte des préférences convexes) de Podczeck [45] (théorème 5.3). Le théorème 3 nous permet de prendre en compte des ensembles de consommations plus généraux que le cône positif. De plus l'hypothèse d'uniforme propreté est plus faible que les hypothèses sur les taux marginaux de substitution présentent dans Jones [28, 29], Ostroy et Zame [43] et Podczeck [45].

Remarque. Nous démontrons un théorème un peu plus général dans le chapitre 4. En particulier, il n'est pas nécessaire de supposer que l'ensemble de production agrégé satisfait la propriété de libre disposition.

Remarque. On peut remplacer l'hypothèse E par l'hypothèse suivante :

Hypothèse (E'). *Il existe* $\overline{v} \in X_\Sigma$ *et* $\overline{u} \in Y_\Sigma$ *tel que* $\omega + \overline{u} - \overline{v} \in \Gamma \cap M(T)_+$.

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Chapitre 1

Quelques résultats sur les correspondances mesurables

1.1 Notations et définitions

On considère $(A, \mathcal{A}, \mu)$ un espace mesuré et (D, d) un espace métrique séparable. Si X est une partie de D, alors l'adhérence de X est notée $\mathrm{cl}\, X$. L'espace mesuré $(A, \mathcal{A}, \mu)$ est dit complet si la σ-algèbre $\mathcal{A}$ contient toutes les parties μ-négligeables [1]. On note $\mathcal{B}$ la σ-algèbre des boréliens sur (D, d). Une fonction $f : A \to D$ est dite mesurable si pour tout ouvert $V \subset D$, $f^{-1}(V) := \{a \in A \mid f(a) \in V\} \in \mathcal{A}$. Une correspondance (ou multifonction) F de A dans D est une application définie sur A à valeurs dans les parties de D, on la note $F : A \twoheadrightarrow D$. Une correspondance $F : A \twoheadrightarrow D$ est mesurable si pour tout ouvert $V \subset D$, l'ensemble $F^{-}(V) := \{a \in A \mid F(a) \cap V \neq \emptyset\} \in \mathcal{A}$. On note $G_F := \{(a, x) \in A \times D \mid x \in F(a)\}$, le graphe de la correspondance F. La correspondance F est de *graphe mesurable* lorsque $G_F \in \mathcal{A} \otimes \mathcal{B}$. Si $F : A \twoheadrightarrow D$ est une correspondance, alors une fonction $f : A \to D$ est une sélection mesurable de F, si f est mesurable et si pour presque tout $a \in A$, $f(a) \in F(a)$. L'ensemble des sélections mesurables de F est noté $S(F)$.

Définition 1.1.1. Une partition $\sigma = (A_i)_{i \in I}$ de A est une *partition mesurable* si pour chaque $i \in I$, l'ensemble A_i est non vide et mesurable, i.e., appartient à $\mathcal{A}$. Un sous ensemble fini A^σ de A est dit *subordonné à la partition* σ s'il existe une famille $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ telle que $A^\sigma = \{a_i \mid i \in I\}$.

1.1.1 Fonctions simples subordonnées à une partition mesurable

Etant donné un couple (σ, A^σ) où $\sigma = (A_i)_{i \in I}$ est une partition mesurable de A, et $A^\sigma = \{a_i \mid i \in I\}$ est un sous ensemble fini subordonné à σ, on considère $\phi(\sigma, A^\sigma)$ l'application qui à chaque fonction mesurable f associe la fonction simple mesurable $\phi(\sigma, A^\sigma)(f)$, définie par

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i) \chi_{A_i} \,,$$

où χ_{A_i} est la fonction indicatrice [2] associée à l'ensemble A_i. Notons que dans la somme définie ci-dessus, un seul des termes peut ne pas être nul. Plus précisément, pour tout $a \in A$, $[\phi(\sigma, A^\sigma)(f)](a) = f(a_i)$ pour $i \in I$ tel que $a \in A_i$.

Définition 1.1.2. Une fonction $s : A \to D$ est appelée *fonction simple subordonnée* à la fonction f, si il existe un couple (σ, A^σ) où σ est une partition mesurable de A, et A^σ est un sous ensemble fini subordonné à σ, tel que $s = \phi(\sigma, A^\sigma)(f)$.

[1] Une partie $N \subset A$ est μ-négligeable, si il existe $E \in \mathcal{A}$ de mesure nulle et contenant N.

[2] C'est à dire, pour tout $a \in A$, $\chi_{A_i}(a) = 1$ si $a \in A_i$ et $\chi_{A_i}(a) = 0$ sinon.

1.1.2 Correspondances simples subordonnées à une partition mesurable

Etant donné un couple (σ, A^σ) où $\sigma = (A_i)_{i\in I}$ est une partition mesurable de A, et $A^\sigma = \{a_i \mid i \in I\}$ est un sous ensemble fini subordonné à σ, on considère $\psi(\sigma, A^\sigma)$, l'application qui à chaque correspondance mesurable $F : A \twoheadrightarrow D$, associe la correspondance simple mesurable $\psi(\sigma, A^\sigma)(F)$, définie par

$$\psi(\sigma, A^\sigma)(F) := \sum_{i\in I} F(a_i)\chi_{A_i}.$$

Définition 1.1.3. Une correspondance $S : A \to D$ est appelée *correspondance simple subordonnée* à la correspondance F si il existe un couple (σ, A^σ) où σ est une partition mesurable de A, et A^σ est un sous-ensemble fini subordonné à σ, tel que $S = \psi(\sigma, A^\sigma)(F)$.

Remarque 1.1.1. Si f est une fonction de A dans D, notons $\{f\}$ la correspondance de A dans D, définie pour tout $a \in A$ par $\{f\}(a) := \{f(a)\}$. On vérifie alors que

$$\psi(\sigma, A^\sigma)(\{f\}) = \{\phi(\sigma, A^\sigma)(f)\}\,.$$

1.1.3 Hyper-espace

Définition 1.1.4. L'ensemble des parties non vides de D est noté $\mathcal{P}^*(D)$. On note τ_{W_d} la topologie de Wijsman sur $\mathcal{P}^*(D)$, i.e., la topologie faible sur $\mathcal{P}^*(D)$ associée à la famille des fonctions distance à un ensemble $(d(x,.))_{x\in D}$. Si $V \subset D$ est un sous ensemble de D, on note $V^- = \{Z \subset D \mid Z \cap V \neq \emptyset\}$, et on note $\mathcal{E}(D)$ la σ-algèbre de Effrös, i.e., la σ-algèbre engendrée par les ensembles de la forme V^-, où V est un ouvert de D.

Hess a démontré dans [10] que, restreintes aux sous ensembles fermés non vides, la σ-algèbre de Effrös $\mathcal{E}(D)$ et les boréliens $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$ de $\mathcal{P}^*(D)$ relativement à la topologie de Wijsman, coïncident. En fait ce résultat reste vrai si on ne se restreint pas aux sous ensembles fermés [3].

Théorème 1.1.1 (Hess).

$$\mathcal{E}(D) = \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}).$$

Démonstration. Si $x \in D$, $\alpha > 0$ et $Z \subset D$, alors on note

$$B(x,\alpha) := \{z \in D \mid d(x,z) < \alpha\} \quad \text{et} \quad \delta_x(Z) := d(x,Z).$$

On vérifie que

$$\delta_x^{-1}([0,\alpha[) = [B(x,\alpha)]^-\,.$$

Ainsi (sans faire usage de l'hypothèse de séparabilité) $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}) \subset \mathcal{E}(D)$. Maintenant, puisque D est séparable, chaque ouvert de D s'écrit comme réunion dénombrable de boules ouvertes. On en déduit que $\mathcal{E}(D) \subset \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$. □

Remarque 1.1.2. Un corollaire du Théorème 1.1.1 est que toute correspondance F de A dans D est mesurable si et seulement si pour tout $x \in D$, la fonction réelle $a \mapsto d(x, F(a))$ est mesurable.

Définition 1.1.5. La semi-métrique de Hausdorff H_d sur $\mathcal{P}^*(D)$ est définie par

$$\forall (A,B) \in \mathcal{P}^*(D) \quad H_d(A,B) := \sup\{|d(x,A) - d(x,B)| \mid x \in D\}.$$

Un sous ensemble C de D est la limite de Hausdorff de la suite $(C_n)_{n\in\mathbb{N}}$ de sous ensembles de D, si

$$\lim_{n\to\infty} H_d(C_n, C) = 0.$$

[3] Notons toutefois que sur $\mathcal{P}^*(D)$, la topologie de Wijsman n'est pas séparée, alors qu'elle l'est sur l'ensemble des parties fermées non vides. Pour plus de précisions, voir Beer [5].

1.2 Discrétisation des fonctions réelles mesurables

On se propose de démontrer que pour une famille dénombrable de fonctions réelles mesurables, il existe une suite de partitions mesurables *approchant* chacune des fonctions.

Théorème 1.2.1. *Soit $\mathcal{F}$ une famille dénombrable de fonctions réelles mesurables. Il existe une suite $(\sigma^n)_{n\in\mathbb{N}}$ de partitions mesurables $\sigma^n = (A_i^n)_{i\in I^n}$ de A, de plus en plus fines, vérifiant les propriétés suivantes.*

(a) *Soit $(A^n)_{n\in\mathbb{N}}$ une suite de sous ensembles finis A^n subordonnés à la partition mesurable σ^n et soit $f \in \mathcal{F}$. Pour chaque $n \in \mathbb{N}$, on définit la fonction simple $f^n := \phi(\sigma^n, A^n)(f)$ subordonnée à f.*

1. *La suite de fonctions $(f^n)_{n\in\mathbb{N}}$ converge simplement vers f.*
2. *Si $f(A)$ est borné alors la suite de fonctions $(f^n)_{n\in\mathbb{N}}$ converge uniformément vers f sur D.*

(b) *Si $\mathcal{G} \subset \mathcal{F}$ est un sous ensemble fini de fonctions intégrables, alors il existe une suite $(A^n)_{n\in\mathbb{N}}$ de sous ensembles finis A^n subordonnés à la partition mesurable σ^n, telle que pour chaque $n \in \mathbb{N}$,*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

En particulier, pour chaque $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Démonstration. Soit $f : A \to \mathbb{R}_+$ une fonction réelle mesurable. Nous allons construire une suite de partitions mesurables dépendant de f. Soit $n \in \mathbb{N}$, on pose $K^n = \{0, \ldots, 2^{2n}\}$ et on définit la partition mesurable $\pi^n(f) = (E_k^n(f))_{k\in K^n}$ où

$$E_k^n(f) = \begin{cases} f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right) & \text{si } k \in \{0, \ldots, 2^{2n} - 1\}, \\ \\ f^{-1}([2^n, +\infty[) & \text{si } k = 2^{2n}. \end{cases}$$

Soit $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ une famille dénombrable de fonctions réelles mesurables. Maintenant pour chaque $n \in \mathbb{N}$, on pose $\mathcal{F}^n := \{f_k \mid 0 \leqslant k \leqslant n\}$ et on définit la partition mesurable σ^n comme suit

$$\sigma^n := (A_i^n)_{i\in I^n} \subset (A_i^n)_{i\in S^n} := \bigvee_{f\in\mathcal{F}^n} [\pi^n(f_+) \vee \pi^n(f_-)],$$

où $I^n := \{i \in S^n \mid A_i^n \neq \emptyset\}$ et $\vee$ est l'opérateur naturel sur les partitions qui à deux partitions π^1 et π^2 associe la partition la moins fine parmi les partitions plus fines que π^1 et π^2.

Nous commençons par démontrer le (a) du théorème 1.2.1. Soit $(A^n)_{n\in\mathbb{N}}$ une suite de sous ensembles finis A^n subordonnés à la partition mesurable σ^n, soit $f \in \mathcal{F}$ et $a \in A$. D'après la construction de σ^n, on peut, sans perte de généralité, supposer que f est positive. Pour n assez grand, $f \in \mathcal{F}^n$ et $f(a) < 2^n$, donc

$$\forall b \in A_i^n \quad |f(b) - f(a)| \leqslant \frac{1}{2^n},$$

où $i \in I^n$ est tel que $a \in A_i^n$. On a donc que $\lim_{n\to\infty} f^n(a) = f(a)$, et cette limite est uniforme si $f(A)$ est borné.

Démontrons maintenant le (b) du théorème 1.2.1. Soit $\mathcal{G} \subset \mathcal{F}$, un sous ensemble fini de fonctions intégrables. Une nouvelle fois, nous pouvons supposer que toutes les fonctions de $\mathcal{G}$ sont positives. Posons $h := \sum_{f\in\mathcal{G}} f$, cette fonction de A dans $\mathbb{R}_+$ est intégrable. Pour chaque $n \in \mathbb{N}$, pour chaque $i \in I^n$, A_i^n est non vide, on peut donc choisir $a_i^n \in A_i^n$ tel que

$$h(a_i^n) \leqslant 1 + \inf\{h(b) \mid b \in A_i^n\}.$$

On a ainsi construit une suite $(A^n)_{n\in\mathbb{N}}$ de sous ensembles finis $A^n := \{a_i^n \mid i \in I^n\}$, subordonnés à la partition mesurable σ^n, telle que pour chaque $f \in \mathcal{G}$, pour tout $n \in \mathbb{N}$,

$$\forall a \in A \quad f^n(a) \leqslant 1 + h(a).$$

En appliquant le théorème de convergence dominée de Lebesgue et (a),

$$\forall f \in \mathcal{G} \quad \lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

□

1.3 Discrétisation des correspondances mesurables

Comme corollaire du théorème 1.2.1, nous proposons de démontrer que, pour une famille dénombrable de correspondances mesurables, il existe une suite de partitions mesurables *approchant* chaque correspondance.

Corollaire 1.3.1. *Soit $\mathcal{F}$ une famille dénombrable de correspondances mesurables de A dans D à valeurs non vides, et soit $\mathcal{G}$ un ensemble fini de fonctions intégrables de A dans $\mathbb{R}$. Il existe une suite $(\sigma^n)_{n\in\mathbb{N}}$ de partitions mesurables $\sigma^n = (A_i^n)_{i\in I^n}$ de A, de plus en plus fines, vérifiant les propriétés suivantes.*

(a) *Soit $(A^n)_{n\in\mathbb{N}}$ une suite d'ensembles finis A^n subordonnés à la partition mesurable σ^n et soit $F \in \mathcal{F}$. Pour chaque $n \in \mathbb{N}$, on définit la correspondance simple $F^n := \psi(\sigma^n, A^n)(F)$ subordonnée à F. Les propriétés suivantes sont alors satisfaites.*

1. *Pour tout $a \in A$, $F(a)$ est la limite de Wijsman de la suite $(F^n(a))_{n\in\mathbb{N}}$, i.e.,*

$$\forall a \in A \quad \forall x \in D \quad \lim_{n\to\infty} d(x, F^n(a)) = d(x, F(a)).$$

2. *Si D est borné alors pour tout $x \in D$ la fonction réelle $d(x, F(.))$ est la limite uniforme de la suite $(d(x, F^n(.)))_{n\in\mathbb{N}}$.*
3. *Si D est totalement borné [4] alors F est la limite de Hausdorff de la suite $(F^n)_{n\in\mathbb{N}}$.*

(b) *Il existe une suite $(A^n)_{n\in\mathbb{N}}$ de sous ensembles finis A^n subordonnés à la partition mesurable σ^n, telle que pour chaque $n \in \mathbb{N}$, si on note $f^n := \phi(\sigma^n, A^n)(f)$ la fonction simple subordonnée à chaque $f \in \mathcal{G}$, alors*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

En particulier, pour chaque $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remarque 1.3.1. La propriété (a1) entraîne en particulier que si $(x^n)_{n\in\mathbb{N}}$ est une suite de D, convergeant vers $x \in D$, alors

$$\forall a \in A \quad \lim_{n\to\infty} d(x^n, F^n(a)) = d(x, F(a)).$$

Ainsi si F est à valeurs non vides fermées, alors (a1) entraîne que

$$\forall a \in A \quad \operatorname{ls} F^n(a) \subset F(a) \subset \operatorname{li} F^n(a).$$ [5]

[4] C'est à dire, pour tout $\varepsilon > 0$ il existe une partie finie $\{x_1, \cdots, x_n\} \subset D$ telle que la collection de boules $B(x_i, \varepsilon) = \{z \in D \mid d(z, x_i) < \varepsilon\}$ recouvre D.

[5] Si $(C_n)_{n\in\mathbb{N}}$ est une suite de sous ensembles de D, la limite (séquentielle) supérieure de $(C_n)_{n\in\mathbb{N}}$, notée $\operatorname{ls} C_n$, est définie par $\operatorname{ls} C_n := \{x \in D \mid x = \lim_{k\to\infty} x_k, \quad x_k \in C_{n(k)}\}$, et la limite (séquentielle) inférieure de $(C_n)_{n\in\mathbb{N}}$, notée $\operatorname{li} C_n$, est définie par $\operatorname{li} C_n := \{x \in D \mid x = \lim_{n\to\infty} x_n, \quad x_n \in C_n\}$.

Démonstration. Si $F : A \twoheadrightarrow D$ est une correspondance, on considère la fonction distance associée $\delta_F : A \times D \to \mathbb{R}_+$ définie par $\delta_F : (a, x) \mapsto \mathrm{d}(x, F(a))$. Soit $F \in \mathcal{F}$, d'après le théorème 1.1.1, F est mesurable si et seulement si, pour chaque $x \in D$, $\delta_F(., x)$ est mesurable. Comme D est séparable, il existe une suite $(x_n)_{n\in\mathbb{N}}$ dense dans D. On pose pour chaque $n \in \mathbb{N}$, $\delta_n^F := \delta_F(., x_n)$. Si $f \in \mathcal{G}$, on considère $|f(.)| : A \to \mathbb{R}_+$ la fonction définie par $a \mapsto |f(a)|$. Posons

$$\mathcal{F}_0 = \{|f(.)| \mid f \in \mathcal{G}\} \cup \bigcup_{F\in\mathcal{F}} \{\delta_n^F \mid n \in \mathbb{N}\} \quad \text{et} \quad \mathcal{G}_0 = \{|f(.)| \mid f \in \mathcal{G}\}.$$

Notons que si F est une correspondance de A dans D, alors pour chaque partition mesurable σ de A, et pour chaque sous ensemble fini A^σ subordonné à σ,

$$\forall x \in L \quad \phi(\sigma, A^\sigma)(d(x, F(.)) = d(x, \psi(\sigma, A^\sigma)(F)(.)).$$

Il suffit maintenant d'appliquer le théorème 1.2.1 à la famille dénombrable $\mathcal{F}_0$ de fonctions mesurables et au sous ensemble fini $\mathcal{G}_0$ de fonctions intégrables. En remarquant que pour chaque $a \in A$, pour chaque $F \in \mathcal{F}$, la fonction $\delta_F(a, .)$ est 1-Lipschitzienne, on obtient le résultat demandé. □

Comme corollaire du corollaire 1.3.1, nous obtenons un résultat de discrétisation des fonctions mesurables.

Corollaire 1.3.2. *Soit $\mathcal{F}$ une famille dénombrable de fonctions mesurables A dans D et soit $\mathcal{G}$ une famille finie de fonctions intégrables de A dans $\mathbb{R}$. Il existe une suite $(\sigma^n)_{n\in\mathbb{N}}$ de partitions mesurables $\sigma^n = (A_i^n)_{i\in I^n}$ de A, de plus en plus fines, vérifiant les propriétés suivantes.*

(a) *Soit $(A^n)_{n\in\mathbb{N}}$ une suite de sous ensembles finis A^n subordonnés à la partition σ^n et soit $f \in \mathcal{F}$. Pour chaque $n \in \mathbb{N}$, on définit la fonction simple $f^n := \phi(\sigma^n, A^n)(f)$ subordonnée à f. Les propriétés suivantes sont alors satisfaites.*

1. *La fonction f est la limite ponctuelle de la suite $(f^n)_{n\in\mathbb{N}}$.*
2. *Si D est totalement borné alors f est la limite uniforme de la suite $(f^n)_{n\in\mathbb{N}}$.*

(b) *Il existe une suite $(A^n)_{n\in\mathbb{N}}$ de sous ensembles finis A^n subordonnés à la partition mesurable σ^n, telle que pour chaque $n \in \mathbb{N}$,*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

En particulier, pour chaque $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remarque 1.3.2. Ce résultat généralise le théorème 4.38 dans Aliprantis et Border [1].

Démonstration. Pour chaque fonction f de A dans D, considérons la correspondance F de A dans D, définie par

$$\forall a \in A \quad F(a) := \{f(a)\},$$

et appliquons le corollaire 1.3.1. □

1.4 Les concepts de mesurabilité pour les préférences

Nous supposons dans cette section que (D, d) est un espace métrique séparable complet.

1.4.1 Mesurabilités des correspondances

Nous rappelons les caractérisations classiques des différentes notions de mesurabilité d'une correspondance. On pourra trouver les preuves des propositions de cette section, dans Castaing et Valadier [6] et Himmelberg [12].

Proposition 1.4.1. *Soit $F : A \twoheadrightarrow D$ une correspondance à valeurs non vides. Les propriétés suivantes sont équivalentes.*

(i) *La correspondance F est mesurable.*

(ii) *Il existe une suite $(f_n)_{n\in\mathbb{N}}$ de sélections mesurables de F telle que pour tout $a \in A$, $F(a) = \mathrm{cl}\{f_n(a) \mid n \in \mathbb{N}\}$.*

(iii) *Pour chaque $x \in D$, la fonction $\delta_F(.,x) : a \mapsto d(x, F(a))$ est mesurable.*

Remarque 1.4.1. Nous avons déjà démontré, comme corollaire du théorème 1.1.1, que (i) est équivalent à (iii), sans utiliser la complétude de (D, d).

Proposition 1.4.2. *Soit $F : A \twoheadrightarrow D$ une correspondance.*

(i) *Si F est à valeurs non vides fermées, alors la mesurabilité de F implique la mesurabilité du graphe de F.*

(ii) *Si $(A, \mathcal{A}, \mu)$ est complet alors la mesurabilité du graphe de F implique la mesurabilité de F.*

(iii) *Si F est à valeurs non vides fermées et si $(A, \mathcal{A}, \mu)$ est complet, la mesurabilité de F est équivalente à la mesurabilité du graphe de F.*

Aumann [3] a démontré (en supposant que $(A, \mathcal{A}, \mu)$ est complet, mais sans supposer que la correspondance est à valeurs fermées.) que si le graphe d'une correspondance est mesurable, alors il existe des sélections mesurables.

Proposition 1.4.3. *Soit F une correspondance de A dans D dont le graphe est mesurable. Si $(A, \mathcal{A}, \mu)$ est complet alors il existe une suite $(z_n)_{n\in\mathbb{N}}$ de sélections mesurables de F, telle que pour tout $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ est dense dans $F(a)$.*

1.4.2 Mesurabilités des préférences

Soit P une correspondance définie sur A à valeurs dans $D \times D$, c'est à dire pour tout $a \in A$, $P(a) \subset D \times D$. Pour chaque fonction $x : A \to D$ la *section supérieure relativement à x* est la correspondance $P_x : A \twoheadrightarrow D$ définie par $a \mapsto \{y \in D \mid (x(a), y) \in P(a)\}$. Symétriquement, pour chaque fonction $y : A \to D$ la *section inférieure relativement à y* est la correspondance $P^y : A \twoheadrightarrow D$ définie par $a \mapsto \{x \in D \mid (x, y(a)) \in P(a)\}$.

Définition 1.4.1. Soit $X : A \twoheadrightarrow D$ une correspondance. Une *correspondance de préférences dans X* est une correspondance P de A dans $D \times D$ vérifiant pour tout $a \in A$, $P(a) \subset X(a) \times X(a)$.

Pour chaque $a \in A$, notons P_a la correspondance [6] de $X(a)$ dans $X(a)$ définie par $x \mapsto \{y \in X(a) \mid (x, y) \in P(a)\}$. Pour chaque $y \in X(a)$ l'image inverse inférieure de y par P_a est notée $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$. Nous rappelons que la correspondance P de préférences (dans X) est dite de graphe mesurable si

$$\{(a, x, y) \in A \times D \times D \mid (x, y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B}.$$

La correspondance P de préférences dans X est dite *Aumann mesurable* si pour toutes sélections mesurables x et y de X,

$$\{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

On peut trouver dans la littérature (Podczeck [15]) d'autres concepts de mesurabilité.

[6] Remarquons que $P(a)$ et le graphe de P_a coïncident.

Définition 1.4.2. La correspondance P de préférences dans X est dite *de graphe mesurable inférieurement*, si pour toute sélection mesurable y de X, la correspondance P^y est de graphe mesurable, c'est à dire

$$\forall y \in S(X) \quad G_{P^y} = \{(a,x) \in A \times D \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}.$$

La correspondance P de préférences dans X est dite *de graphe mesurable supérieurement*, si pour toute sélection x de X, la correspondance P_x est de graphe mesurable, c'est à dire,

$$\forall x \in S(X) \quad G_{P_x} = \{(a,y) \in A \times D \mid (x(a), y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}.$$

Nous proposons de comparer ces trois concepts de mesurabilité des préférences.

Proposition 1.4.4. *Soit P une correspondance de préférences dans X. Nous supposons que $(A, \mathcal{A}, \mu)$ est complet et que la correspondance X est de graphe mesurable.*

Si P est de graphe mesurable, alors P est de graphe mesurable supérieurement et inférieurement. De plus si P est de graphe mesurable supérieurement ou inférieurement, alors P est Aumann mesurable.

Démonstration. C'est une conséquence directe du théorème de Projection dans Castaing et Valadier [6]. □

Sous des hypothèses supplémentaires on démontre les réciproques.

Proposition 1.4.5. *Soit P une correspondance de préférences dans X. Nous supposons que $(A, \mathcal{A}, \mu)$ est complet et la correspondance X est de graphe mesurable. De plus nous supposons que pour presque tout $a \in A$, $X(a)$ est fermé et connexe, $P(a)$ est une relation binaire sur $X(a)$, irréflexive et transitive, et pour chaque $x \in X(a)$, $P_a(x)$ et $P_a^{-1}(x)$ sont ouverts dans $X(a)$.*

Lorsque l'une des propriétés suivantes est satisfaite,

1. *pour presque tout $a \in A$, $X(a) = (\mathbb{R}_+)^\ell$ où [7] $D = \mathbb{R}^\ell$ et $P(a)$ est strictement monotone [8],*
2. *pour presque tout $a \in A$, $P(a)$ est négativement transitive,*

si P est Aumann mesurables, alors P est de graphe mesurable supérieurement et inférieurement. De plus si P est de graphe mesurable supérieurement et inférieurement, alors P est de graphe mesurable.

Démonstration. Supposons que P est Aumann mesurable. Nous distinguons deux cas. Sous la propriété 1, $(\mathbb{Q}_+)^\ell$ est dense dans $X(a)$ pour tout $a \in A$, donc si $(x,y) \in P(a)$ alors il existe $r \in (\mathbb{Q}_+)^\ell$ tel que $(x,r) \in P(a)$ et $r < y$. Ainsi, si $x \in S(X)$ est une sélection mesurable de X, alors

$$G_{P_x} = \bigcup_{r \in \mathbb{Q}_+^\ell} \left(\{(a \in A \mid (x(a), r) \in P(a)\} \times (\mathbb{R}_+)^\ell \right) \cap (A \times \{y \in D \mid r < y\})$$

et $G_{P_x} \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$. De même on peut démontrer que $G_{P^x} \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$.

Sous la propriété 2, pour démontrer que les sections inférieures et supérieures sont de graphe mesurable, nous nous inspirons fortement de la preuve du lemme dans l'appendice de Podczeck [15]. Le graphe de X est mesurable, donc, d'après la proposition 1.4.2, X possède une représentation de Castaing, c'est à dire, il existe une suite $(h_i)_{i \in \mathbb{N}}$ de sélections mesurables de X, telle que pour tout $a \in A$, $X(a) = \mathrm{cl}\{h_i(a) \mid i \in \mathbb{N}\}$. Soit $x \in S(X)$ une sélection mesurable de X. Considérons un agent $a \in A$ et $y \in X(a)$. Si $(x(a), y) \in P(a)$, alors en suivant les arguments de Debreu [8], il existe $i \in \mathbb{N}$ tel que $(x(a), h_i(a)) \in P(a)$ et $(h_i(a), y) \in P(a)$. En utilisant les hypothèses de continuité de $P(a)$, pour tout $n \in \mathbb{N}$, il existe $j \in \mathbb{N}$ tel que $d(y, h_j(a)) \leqslant 1/n$ et $(h_i(a), h_j(a)) \in P(a)$. Inversement, si pour un indice $i \in \mathbb{N}$, $(x(a), h_i(a)) \in P(a)$ et pour pour chaque $n \in \mathbb{N}$, il existe $j \in \mathbb{N}$ tel que $d(y, h_j(a)) \leqslant 1/n$ et $(h_i(a), h_j(a)) \in P(a)$, alors $y \in \mathrm{cl}\, P_a(h_i(a)) \subset P_a(x(a))$. Ainsi

$$G_{P_x} = G_X \cap \bigcup_{i \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} [(A(i,j) \times D) \cap \{(a,y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\}] \ ,$$

[7] Pour un entier $\ell \in \mathbb{N}$.

[8] C'est à dire, pour tout $x \in X(a)$, pour tout $m \in (\mathbb{R}_+)^\ell$, $x + m \in P_a(x) \cup \{x\}$.

où
$$A(i,j) = \{a \in A \mid (x(a), h_i(a)) \in P(a)\} \cap \{a \in A \mid (h_i(a), h_j(a)) \in P(a)\}.$$
Comme P est Aumann mesurable, pour chaque $(i,j) \in \mathbb{N}^2$, $A(i,j) \in \mathcal{A}$. De plus, d'après [6] où [12], pour chaque $(j,n) \in \mathbb{N}^2$, $\{(a,y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\} \in \mathcal{A} \times \mathcal{B}$, et les sections supérieures de P sont de graphe mesurable. De même on peut démontrer que les sections inférieures de P sont de graphe mesurable.

Supposons maintenant que les sections inférieures et supérieures de P sont de graphe mesurable. Soit $(a,x,y) \in G_P$, c'est à dire $(x,y) \in P(a)$. Nous distinguons deux cas. Sous la propriété 2, il existe $i \in \mathbb{N}$ tel que
$$(x, h_i(a)) \in P(a) \quad \text{and} \quad (h_i(a), y) \in P(a).$$
Comme $P(a)$ est une relation transitive, la réciproque est vérifiée, et
$$G_P = \bigcup_{i \in \mathbb{N}} \left\{(a,x,y) \in A \times D \times D \mid (a,x) \in G_{P(.,h_i(.))} \quad \text{and} \quad (a,y) \in G_{P^{-1}(.,h_i(.))}\right\}.$$
Ainsi le graphe de P est mesurable.

Sous la propriété 1, il existe $r \in (\mathbb{Q}_+)^\ell$ tel que $(x,r) \in P(a)$ et $r < y$. Comme les préférences sont monotones, la réciproque est vérifiée et
$$G_P = \bigcup_{r \in (\mathbb{Q}_+)^\ell} \{(a,x,y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell \mid (x,r) \in P(a)\} \times \{(a,x,y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell \mid r < y\}.$$
Ainsi le graphe de P est mesurable. □

Rappelons qu'une correspondance P_a est semi-continue inférieurement si pour tout ouvert $V \subset D$, $\{x \in X(a) \mid P_a(x) \cap V \neq \emptyset\}$ est ouvert dans $X(a)$.

Nous introduisons une notion de mesurabilité des préférences proche de la notion de semi-continuité inférieure.

Définition 1.4.3. La correspondance de préférences P dans X est de *graphe semi-mesurable inférieurement* si pour toute correspondance de graphe mesurable $V : A \twoheadrightarrow D$ à valeurs ouvertes, l'ensemble suivant est mesurable
$$\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} \in \mathcal{A} \times \mathcal{B}.$$

Nous proposons de comparer ce concept de mesurabilité avec les autres concepts utilisés dans la littérature.

Proposition 1.4.6. *Soit P une correspondance de préférences dans X. Nous supposons que $(A, \mathcal{A}, \mu)$ est complet et que X est de graphe mesurable.*

(i) *Si P est de graphe mesurable alors P est de graphe semi-mesurable inférieurement.*

(ii) *Supposons pour presque tout $a \in A$, pour tout $x \in X(a)$, $P_a(x)$ est ouvert dans $X(a)$. Si P est de graphe mesurable inférieurement, alors P est de graphe semi-mesurable inférieurement.*

(iii) *Supposons pour presque tout $a \in A$, pour tout $x \in X(a)$, $P_a(x)$ est fermé dans $X(a)$. Si P est de graphe semi-mesurable inférieurement, alors P est de graphe mesurable inférieurement.*

Démonstration. La propriété (i) est une conséquence directe du théorème de Projection dans Castaing et Valadier [6]. En effet,
$$\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} = \pi\left[G_P \cap \{(a,x,y) \in A \times D \times D \mid y \in V(a)\}\right],$$
où $\pi : A \times D \times D \to A \times D$ est la projection $(a,x,y) \mapsto (a,x)$.

Supposons maintenant que la correspondance P est de graphe mesurable inférieurement et que pour presque tout $a \in A$, pour tout $x \in X(a)$, $P_a(x)$ est ouvert dans $X(a)$. Soit $(a,x) \in G_X$ tel que $P_a(x) \cap V(a) \neq \emptyset$. D'après la proposition 1.4.3, il existe une suite $(z_n)_{n \in \mathbb{N}}$ de sélections mesurables de

X, tel que pour tout $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ est dense dans $X(a)$. L'ensemble $P_a(x) \cap V(a)$ est ouvert dans $X(a)$, ainsi il existe $n \in \mathbb{N}$ tel que $z_n(a) \in P_a(x) \cap V(a)$. La réciproque est vérifiée et

$$\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \left[G_{P^{z_n}} \cap (\{a \in A \mid z_n(a) \in V(a)\} \times D)\right].$$

Ainsi P est de graphe semi-mesurable inférieurement.

Supposons maintenant que la correspondance P est de graphe semi-mesurable inférieurement et que pour presque tout $a \in A$, pour tout $x \in X(a)$, $P_a(x)$ est fermé dans $X(a)$. Soit $y \in S(X)$ une sélection mesurable de X. Soit $(a,x) \in G_{P^y}$, c'est à dire, $x \in X(a)$ et $y(a) \in P_a(x)$. Soit $n \in \mathbb{N}$, on pose $V_n(a) = \{z \in D \mid d(z, y(a)) < 1/(n+1)\}$. Alors pour tout $n \in \mathbb{N}$, $P_a(x) \cap V_n(a) \neq \emptyset$. Réciproquement, si pour tout $n \in \mathbb{N}$, $P_a(x) \cap V_n(a) \neq \emptyset$, alors $y(a)$ est adhérent à $P_a(x)$. Comme $P_a(x)$ est fermé dans $X(a)$, on a $(a,x) \in G_{P^y}$. Ainsi

$$G_{P^y} = \bigcap_{n\in\mathbb{N}} \{(a,x) \in G_X \mid P_a(x) \cap V_n(a) \neq \emptyset\}.$$

Et la correspondance P est de graphe mesurable inférieurement. $\square$

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Chapitre 2

Existence d'équilibres avec un espace mesuré d'agents et des préférences non ordonnées

Résumé

Nous proposons une nouvelle approche pour démontrer l'existence d'un équilibre de Walras pour des économies avec un espace mesuré d'agents et un espace des biens de dimension finie. Notre approche, basée sur la discrétisation des correspondances (ou multifonctions) mesurables, nous permet de démontrer l'existence d'un équilibre aussi bien pour des économies avec des préférences non ordonnées mais convexes, que pour des économies avec des préférences partiellement ordonnées mais non convexes. Notre résultat d'existence généralise les resultats de Aumann [4], Schmeidler [30] et Hildenbrand [21].

Mots-clés : *Espace mesuré d'agents, préférences non ordonnées mais convexes, préférences ordonnées mais non convexes et discrétisation des correspondances mesurables*

Existence of equilibria for economies with a measure space of agents and non-ordered preferences

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Abstract

A new approach is proposed to prove the existence of a Walrasian equilibrium for production economies with a measure space of agents and finitely many commodities. The new approach, based on the discretization of measurable correspondences, allows us to provide an existence result for economies with non-ordered but convex preferences as well as for economies with partially ordered (possibly incomplete) but non-convex preferences. This paper generalizes results of Aumann [4], Schmeidler [30] and Hildenbrand [21].

Keywords : *Measure space of agents, non-ordered but convex preferences, partially ordered but non-convex preferences and discretization of measurable correspondences.*

2.1 Introduction

Aumann [4] and Hildenbrand [21] provide existence results of Walrasian equilibria for exchange and production economies with a measure space of agents and ordered preferences. In the framework of strictly monotone preferences, the Main Theorem in Schmeidler [30] dispenses with completeness of preferences. In recent years attempts (e.g. in [25]) were made to generalize these results to economies with externalities in consumption. In Balder [7], it is shown that the usual conditions used for these attempts force the *preferred to* correspondence to be empty-valued almost everywhere on the non-atomic part of the measure space of agents, rendering these attempts pointless.

Following a *discretization* approach, we provide in this paper an existence result for both non-ordered (but without externalities in consumption) and partially ordered (possibly incomplete) preferences. For economies with non-ordered preferences, we can not dispense with a convexity assumption on preferences. Indeed, we provide a simple counterexample of a continuum economy with non-transitive preferences, satisfying all usual assumptions except convexity, and for which no Walrasian equilibrium exists. For economies with partially ordered preferences, our result generalizes the Main Theorem in Aumann [4], the Main Theorem in Schmeidler [30] and Theorems 1 and 2 in Hildenbrand [21].

The *discretization* approach proposed in this paper consists of considering an economy with a measure space of agents as the *limit* of a sequence of economies with a finite, but larger and larger, set of agents. We construct a sequence of partitions of the measure space depending on the measurable characteristics of the economy. To each partition we define a *subordinated simple* economy. Each *simple* economy will be identified as an economy with a finite set of agents, and applying a classical equilibria existence result for economies with finitely many agents, we get a sequence of equilibria which will converge to a quasi-equilibrium for the initial economy.

The paper is organized as follows. In Section 2.2, we set out the main definitions and notations. In Section 2.3 we define the model of production economies with a measure space of agents, we introduce the concepts of equilibria, we give the list of assumptions that economies will be required to satisfy and finally, we present an existence result (Theorem 2.3.1) for free-disposal economies and an existence result (Corollary 2.3.1) for economies with strictly monotone preferences. Section 2.4 is devoted to the mathematical *discretization* of measurable correspondences. The proof of the main existence result (Theorem 2.3.1) is given in Section 2.5. The existence result for economies with finitely many agents is provided in Appendix A and Appendix B is devoted to mathematical auxiliary results about measurability and integration of correspondences.

2.2 Notations and definitions

Let $\mathbb{L}$ be a finite dimensional vector space induced with its natural topology. The dual of $\mathbb{L}$ is noted $\mathbb{L}^*$ and the natural dual pairing $\langle \mathbb{L}^*, \mathbb{L}\rangle$ is defined by $\langle p, x\rangle = p(x)$ for each $(p,x) \in \mathbb{L}^* \times \mathbb{L}$. Let $C \subset \mathbb{L}$ be a pointed convex cone[1]. The partial order induced[2] by C is noted $\geqslant$. We note $\mathbb{L}_+$ the positive cone $\{x \in \mathbb{L} \mid x \geqslant 0\}$. If $x \in \mathbb{L}$ then we note $x > 0$ ($x \gg 0$) if $x \geqslant 0$ and $x \neq 0$ (resp. x is an interior point of C). In the dual space $\mathbb{L}^*$ we let ${\mathbb{L}^*}_+ = \{p \in \mathbb{L}^* \mid \forall c \in X \quad p\,(c) \geqslant 0\}$ and we note $p \geqslant 0$ ($p > 0$) if $p \in {\mathbb{L}^*}_+$ (resp. $p \in {\mathbb{L}^*}_+$ and $p \neq 0$). A strictly positive functional, written $p \gg 0$ is a positive functional satisfying $p\,(x) > 0$ for all $0 < x \in \mathbb{L}$. If $X \subset \mathbb{L}$ is a subset, then the interior of X is noted $\operatorname{int} X$, the closure of X is noted $\operatorname{cl} X$. If $p \in \mathbb{L}^*$ then we let $p(X) = \{p(x) \mid x \in X\}$ and if $Y \subset \mathbb{L}$ then $p(X) \geqslant p(Y)$ means [if $(x,y) \in X \times Y$ then $p(x) \geqslant p(y)$]. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{L}$, the *sequential upper limit* of $(C_n)_{n\in\mathbb{N}}$, noted $\operatorname{ls} C_n$, is defined by

$$\operatorname{ls} C_n := \left\{ x \in \mathbb{L} \;\middle|\; x = \lim_{k\to\infty} x_k\,, \quad x_k \in C_{n(k)} \right\}.$$

The convex hull of X is noted $\operatorname{co} X$ and the closed convex hull of X is noted $\overline{\operatorname{co}}\, X$. If X is convex then we let $A(X) = \{v \in \mathbb{L} \mid X + \{v\} \subset X\}$ be the asymptotic cone of X. Note that if X is closed convex, then $A(X)$ is the set of vectors $v \in \mathbb{L}$ such that $v = \lim_{n\to\infty} \lambda_n u_n$ where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence decreasing to 0 and $(u_n)_{n\in\mathbb{N}}$ is a sequence in X.

We consider $(A, \mathcal{A}, \mu)$ a finite measure space, that is, A is a set, $\mathcal{A}$ is a σ-algebra of subsets of A and μ is a finite measure on $\mathcal{A}$. The measure space $(A, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all μ-negligible[3] subsets of A.

Let (D, d) be a separable metric space. The σ-algebra of Borel subsets of D is noted $\mathcal{B}(D)$. A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for every open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a,x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$. When $D \subset \mathbb{L}$ the set of integrable selections of F is noted $S^1(F)$ and we note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a)d\mu(a) := \left\{ v \in D \;\middle|\; \exists x \in S^1(F) \quad v = \int_A x(a)d\mu(a) \right\}$.

Let X be a space and $P \subset X \times X$ be a binary relation on X. The relation P is irreflexive if $(x,x) \notin P$, for all $x \in X$. The relation P is transitive if $[(x,y) \in P$ and $(y,z) \in P]$ implies $(x,z) \in P$, for all $(x,y,z) \in X^3$. The relation P is negatively transitive if $[(x,y) \notin P$ and $(y,z) \notin P]$ implies $(x,z) \notin P$, for all $(x,y,z) \in X^3$. The relation P is a partial order if it is irreflexive and transitive. The relation P is an order if it is irreflexive, transitive and negatively transitive. When P is an order, it is usually noted $\succ$ and $X^2 \setminus P$ is noted $\preceq$. Note that when P is an order, then $\preceq$ is transitive, reflexive ($x \preceq x$ for all $x \in X$) and complete (for all $(x,y) \in X^2$ either $x \preceq y$ or $y \preceq x$).

2.3 The model, the equilibrium concepts and the assumptions

2.3.1 The Model

We consider a finite dimensional vector space $\mathbb{L}$, a complete measure space $(A, \mathcal{A}, \mu)$, a function e from A to $\mathbb{L}$, two correspondences X and Y from A into $\mathbb{L}$ and a correspondence of preference relations P in X, that is, P is a correspondence from A into $\mathbb{L} \times \mathbb{L}$ such that for all $a \in A$, $P(a) \subset X(a) \times X(a)$ and $P(a)$ is irreflexive.

An economy $\mathcal{E}$ is a list

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L}\rangle, (X, Y, P, e)).$$

The commodity space is represented by $\mathbb{L}$ and the natural dual pairing $\langle \mathbb{L}^*, \mathbb{L}\rangle$ is interpreted as the *price-commodity* pairing.

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[1]That is C is a cone: $\alpha C \subset C$ for all $\alpha \geqslant 0$, C is convex: $C + C \subset C$ and C is pointed: $C \cap (-C) = \{0\}$.

[2]That is for all $(x,y) \in \mathbb{L}^2$, $x \geqslant y$ whenever $x - y \in C$.

[3]A set N is μ-negligible if there exists $E \in \mathcal{A}$ such that $N \subset E$ and $\mu(E) = 0$.

The set of agents (or consumers) is represented by A, the set $\mathcal{A}$ represents the set of admissible coalitions, and the number $\mu(E)$ represents the fraction of consumers which are in the coalition $E \in \mathcal{A}$.

For each agent $a \in A$, the consumption set is represented by $X(a) \subset \mathbb{L}$ and the preference relation is represented by $P(a)$. We define the correspondence [4] $P_a : X(a) \twoheadrightarrow X(a)$ by $P_a(x) = \{x' \in X(a) \mid (x, x') \in P(a)\}$. In particular, if $x \in X(a)$ is a consumption bundle, the set $P_a(x)$ is the set of consumption bundles strictly preferred to x by the agent a. The set of consumption allocations (or plans) of the economy is the set $\mathrm{S}^1(X)$ of integrable selections of X. The aggregate consumption set X_Σ is defined by

$$X_\Sigma := \int_A X(a)d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in \mathrm{S}^1(X) \quad v = \int_A x(a)d\mu(a) \right\}.$$

The initial endowment of the consumer $a \in A$ is represented by the commodity bundle $e(a) \in \mathbb{L}$. We assume that the function $e : A \to \mathbb{L}$ is an integrable function and we note $\omega := \int_A e(a)d\mu(a)$ the aggregate initial endowment. The production possibilities available to the consumer $a \in A$ are represented by the set $Y(a) \subset \mathbb{L}$. The set of production allocations (or plans) of the economy is the set $\mathrm{S}^1(Y)$ of integrable selections of Y. The aggregate production set Y_Σ is defined by

$$Y_\Sigma := \int_A Y(a)d\mu(a) = \left\{ u \in \mathbb{L} \;\middle|\; \exists y \in \mathrm{S}^1(Y) \quad u = \int_A y(a)d\mu(a) \right\}.$$

2.3.2 The Equilibrium Concepts

Definition 2.3.1. A *Walrasian equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ such that $p^* \neq 0$ and satisfying the following properties.

(a) For almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a)) \quad \text{and} \quad x \in P_a(x^*(a)) \Longrightarrow p^*(x) > p^*(x^*(a)).$$

(b) For almost every $a \in A$,

$$y \in Y(a) \Longrightarrow p^*(y) \leqslant p^*(y^*(a)).$$

(c)

$$\int_A x^*(a)d\mu(a) = \int_A e(a)d\mu(a) + \int_A y^*(a)d\mu(a).$$

A *Walrasian quasi-equilibrium* of an economy $\mathcal{E}$ is an element $(x^*, y^*, p^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ such that $p^* \neq 0$ and which satisfies the conditions (b) and (c) together with

(a') for almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a)) \quad \text{and} \quad x \in P_a(x^*(a)) \Longrightarrow p^*(x) \geqslant p^*(x^*(a)).$$

A Walrasian equilibrium of an economy $\mathcal{E}$ is clearly a Walrasian quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, classical conditions on $\mathcal{E}$ under which a Walrasian quasi-equilibrium is in fact a Walrasian equilibrium.

Remark 2.3.1. Every quasi-equilibrium (x^*, y^*, p^*) of a production economy $\mathcal{E}$ is an equilibrium if we assume that, for almost every agent $a \in A$, $X(a)$ is convex, the strict-preferred set $P_a(x^*(a))$ is open in $X(a)$ and

$$\inf p^*(X(a)) < p^*(e(a)) + \sup p^*(Y(a)).$$

In particular, if $p^* > 0$ then the last condition is automatically valid if for almost every agent $a \in A$,

$$\Big(\{e(a)\} + Y(a) - X(a)\Big) \cap \operatorname{int} \mathbb{L}_+ \neq \emptyset.$$

[4] Note that the binary relation $P(a)$ coincide with the graph of the correspondence P_a.

The model of production economies defined above encompasses the two models presented in Hildenbrand [21].

In a *private ownership economy* $\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, P, e), (Y_j, \theta_j)_{j \in J})$, the production sector is represented by a finite set J of firms with production sets $(Y_j)_{j\in J}$, where for every $j \in J$, $Y_j \subset \mathbb{L}$. The profit made by the firm $j \in J$ is distributed among the consumers following a share function $\theta_j : A \to \mathbb{R}_+$. The share functions are supposed to be μ-integrable and to satisfy for each $j \in J$, $\int_A \theta_j(a) d\mu(a) = 1$. If we let for each $a \in A$,

$$Y(a) := \sum_{j \in J} \theta_j(a) \overline{\text{co}}\, Y_j$$

then we define a production economy $\mathcal{E}' := ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, Y, P, e))$. If the production sector of the private ownership economy satisfies $\sum_{j\in J} Y_j$ is closed convex, then for all $p \in \mathbb{L}^*$ and for almost every $a \in A$,

$$\int_A Y(a) d\mu(a) = \sum_{j \in J} Y_j \quad \text{and} \quad \sup p\,(Y(a)) = \sum_{j\in J} \theta_j(a) \sup p\,(Y_j)\,.$$

It follows that the notion (defined in Hildenbrand [21]) of Walrasian equilibrium for the private ownership economy $\mathcal{E}$, and the notion (defined in this paper) of Walrasian equilibrium for the associated production economy $\mathcal{E}'$, coincide.

In a *coalition production economy* $\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, P, e), \mathbf{Y})$, the production sector is defined for every coalition $E \in \mathcal{A}$ by a production set $\mathbf{Y}(E) \subset \mathbb{L}$. In the framework of Hildenbrand [21], the correspondence $\mathbf{Y} : \mathcal{A} \twoheadrightarrow \mathbb{L}$ is supposed to be countably additive and to admit a Radon-Nikodym derivative. If we let $Y : A \twoheadrightarrow \mathbb{L}$ be a Radon-Nikodym derivative of $\mathbf{Y}$ then we define a production economy $\mathcal{E}' = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, Y, P, e))$. If $\mathbf{Y}(A)$ is closed convex, then for every $p \in \mathbb{L}^*$ and for every coalition $E \in \mathcal{A}$,

$$\sup p\,(\mathbf{Y}(E)) = \int_E \sup p\,(Y(a))\, d\mu(a).$$

Hence the notion (defined in Hildenbrand [21]) of Walrasian equilibrium for the coalition production economy $\mathcal{E}$, and the notion (defined in this paper) of Walrasian equilibrium for the associated production economy $\mathcal{E}'$, coincide.

2.3.3 The Assumptions

We present the list of assumptions that economies will be required to satisfy. We suppose that $\mathbb{L}$ is endowed with a linear order defined by a pointed closed convex cone $\mathbb{L}_+$. On the consumption side we consider both non-ordered but convex preferences (Assumption C_n) and partially ordered (possibly incomplete) but non-convex preferences (Assumption C_p).

Assumption (C_n). ***[non-ordered but convex]*** *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is continuous, that is, for all $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ [5] are open in $X(a)$,*

(ii) *the preference relation $P(a)$ is convex, that is, the consumption set $X(a)$ is convex and for each bundle $x \in X(a)$, $x \notin \text{co}\, P_a(x)$.*

Assumption (C_p). ***[partially ordered but non-convex]*** *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is continuous,*

(ii) *if a belongs to the non-atomic [6] part of $(A, \mathcal{A}, \mu)$ then $P(a)$ is partially ordered, and if a belongs to an atom of $(A, \mathcal{A}, \mu)$, then the preference relation $P(a)$ is convex.*

[5] For each $y \in X(a)$, $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$.

[6] An element $E \in \mathcal{A}$ is an atom of $(A, \mathcal{A}, \mu)$ if $\mu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\mu(B) = 0$ or $\mu(E \setminus B) = 0$.

Remark 2.3.2. In the frameworks of Aumann [4], Hildenbrand [21], Schmeidler [30] and Cornet and Topuzu [11], Assumption C_p is valid. In particular, in [11], it is supposed that for each agent a in the atomic part of $(A, \mathcal{A}, \mu)$, $P(a)$ is partially ordered and for each bundle $x \in X(a)$, $X(a) \setminus P_a^{-1}(x)$ is convex. This implies that for each agent a in the atomic part of $(A, \mathcal{A}, \mu)$, $P(a)$ is convex.

Remark 2.3.3. In general, Assumptions C_n and C_p are not comparable but if for almost every agent $a \in A$, the preference relation $P(a)$ is convex, then Assumption C_p implies Assumption C_n.

Assumption (C). ***[Consumption side]*** *Assumption C_p or Assumption C_n is satisfied.*

Assumption (M). ***[Measurability]*** *The correspondences X and Y are graph measurable, that is,*

$$\{(a,x) \in A \times \mathbb{L} \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}) \quad \textit{and} \quad \{(a,y) \in A \times \mathbb{L} \mid y \in Y(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L})$$

and the correspondence of preferences P is graph measurable, that is,

$$\{(a,x,y) \in A \times \mathbb{L} \times \mathbb{L} \mid (x,y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}) \otimes \mathcal{B}(\mathbb{L}).$$

Remark 2.3.4. Under Assumption C, if preferences are ordered, following Proposition 2.7.5, we can replace in Assumption M, the graph measurability of P by the Aumann measurability of preferences, that is

$$\forall x, y \in S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

Remark 2.3.5. In the framework of Aumann [4] and Schmeidler [30], it is assumed that preferences are *Aumann measurable.* Applying Proposition 2.7.5, the preferences P are then graph measurable and Assumption M is valid.

Assumption (P). ***[Production side]*** *The aggregate production set Y_Σ is a non-empty closed convex subset of $\mathbb{L}$.*

Remark 2.3.6. In the literature dealing with *private ownership economies* it is assumed that for every $j \in J$, Y_j is non-empty. It obviously implies that Y_Σ is non-empty. In the literature dealing with *coalitional production economies*, e.g. in Hildenbrand [21], it is assumed that inaction is possible, that is, for almost every $a \in A$, $0 \in Y(a)$. Once again this assumption implies that Y_Σ is non-empty.

If we let $\tilde{Y} : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined for all $a \in A$ by

$$\tilde{Y}(a) := \mathrm{cl}\left(\overline{\mathrm{co}}\, Y(a) + A(Y_\Sigma)\right),$$

then following Proposition 2.7.7, $\tilde{Y}$ satisfies Assumption P, and the economy $\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, Y, P, e))$ has a Walrasian (quasi-) equilibrium if and only if the economy $\tilde{\mathcal{E}} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X, \tilde{Y}, P, e))$ has a Walrasian (resp. quasi-) equilibrium.

Assumption (S). ***[Survival]*** *For almost every $a \in A$,*

$$X(a) \cap \left(\{e(a)\} + \tilde{Y}(a)\right) \neq \emptyset.$$

Remark 2.3.7. Assumption S means that we need compatibility between individual needs and resources. In [21], Hildenbrand supposed that for almost every agent $a \in A$, $0 \in Y(a)$ and $X(a) \cap (\{e(a)\} + A(Y_\Sigma)) \neq \emptyset$. Yamazaki in [34] proposed a different survival assumption.

Assumption (B). ***[Bounded]*** *The consumption set correspondence X is integrably bounded from below* [7] *and the set of free-production $Y_\Sigma \cap \mathbb{L}_+$ is bounded.*

Remark 2.3.8. In Hildenbrand [21], the commodity space is $\mathbb{L} = \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$ and since it is Fatou's lemma of Schmeidler [31] that is used, the positive cone is supposed to be $\mathbb{L}_+ = (\mathbb{R}_+)^\ell$. Here we apply the recent Fatou's lemma of Cornet and Topuzu [31] (Theorem 2.7.2), which allows us to consider a more general pointed convex cone. Note that $\mathbb{L}_+$ is not supposed to have an interior point. The boundedness assumption of the free-production set is a weaker assumption than the corresponding one in [21]. Indeed, Hildenbrand assumed that the aggregate production set has no free-production, that is, $Y_\Sigma \cap \mathbb{L}_+ = \{0\}$.

[7] That is there exists an integrable function $\underline{x}$ from A to $\mathbb{L}$ such that for a.e. $a \in A$, $X(a) \subset \{\underline{x}(a)\} + \mathbb{L}_+$.

Assumption (LNS). ***[Local Non Satiation]*** *For almost every agent $a \in A$, for all bundle $x \in X(a)$, $x \in \overline{\mathrm{co}}\, P_a(x)$.*

Remark 2.3.9. Hildenbrand in [21] assumed a stronger assumption, which is, for a.e. $a \in A$, for all $a \in X(a)$, $x \in \mathrm{cl}\, P_a(x)$.

2.3.4 Existence of equilibria for free-disposal economies

Assumption (FD). ***[Free Disposal]*** *One of the two following properties holds.*

(a) *The aggregate production set is free-disposal, that is, $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$.*

(b) *The preferences are weakly monotone, that is, for almost every agent $a \in A$, $X(a) + \mathbb{L}_+ \subset X(a)$ and for all $(x, y) \in X(a) \times X(a)$, $y \geqslant x \Rightarrow P_a(y) \subset P_a(x)$.*

Remark 2.3.10. If preferences are supposed to be strictly monotone (Assumption MON in the next subsection) and transitive, then the condition (b) in Assumption FD is automatically valid.

In order to prove that a quasi-equilibrium of $\mathcal{E}$ is in fact an equilibrium, the economy will be required to satisfy the following assumption.

Assumption (SS). ***[Strong Survival]*** *For almost every agent $a \in A$, there exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in int\mathbb{L}_+$ and such that $X(a)$ is star-shaped* [8] *about $x^0(a)$.*

Remark 2.3.11. In [21], Hildenbrand assumed that for almost every $a \in A$, $X(a)$ is convex (and thus star-shaped about each point), $0 \in Y(a)$ and $(\{e(a)\} + \mathrm{int}\, A(Y_\Sigma)) \cap X(a) \neq \emptyset$. This assumption obviously implies Assumption SS. The consumption $X(a)$ need not to be convex in order to satisfy Assumption SS. For example, if we take $X(a) = \{(x, y) \in \mathbb{R}^2_+ \mid x = 0 \text{ or } y = 0\}$ and $e(a) = (1, 1)$ for all $a \in A$, and $Y_j = -\mathbb{R}^2_+$ for all $j \in J$, then assumption SS is satisfied.

We are now ready to state the first existence result.

Theorem 2.3.1. *If an economy $\mathcal{E}$ satisfies Assumptions **C**, **M**, **P**, **S**, **B**, **LNS** and **FD**, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* > 0$. If moreover $\mathcal{E}$ satisfies **SS** then (x^*, y^*, p^*) is a Walrasian equilibrium of $\mathcal{E}$.*

Remark 2.3.12. This equilibrium existence result improves Theorem 1 and 2 in Hildenbrand [21]. Indeed, Assumptions C_p, M, P, S, B, LNS, FD and SS of Theorem 2.3.1 are implied by those used in [21]. More precisely, we only require that preferences are partially ordered. We do not need to suppose, as in Hildenbrand [21], that preferences are ordered. Moreover, to prove the existence of a quasi-equilibrium, we do not assume that consumption sets are convex on the non-atomic part of $(A, \mathcal{A}, \mu)$. Neither do we need to suppose that the aggregate production set Y_Σ satisfies an irreversibility property $Y_\Sigma \cap (-Y_\Sigma) = \{0\}$. Instead of supposing impossibility of free-production $Y_\Sigma \cap \mathbb{L}_+ = \{0\}$, we only suppose that the set of free-production is bounded. We replace possibility of inaction, that is, for almost every $a \in A$, $0 \in Y(a)$, by the weaker assumption that the aggregate production set is non-empty. Moreover Fatou's Lemma of Cornet and Topuzu [11] allows us to deal with a more general positive cone than $(\mathbb{R}_+)^\ell$ when $\mathbb{L} = \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$.

Aumann in [4] for exchange economies and Hildenbrand in [21] for production economies proved that for continuum economies, that is, economies with a non-atomic measure space of agents, the convex assumption on ordered preferences is not needed to prove the existence of a Walrasian equilibrium. But in Theorem 2.3.1, when preferences are possibly non-ordered (Assumption C_n) they are assumed to satisfy a convexity property. We provide hereafter an example of a production economy satisfying all assumptions of Theorem 2.3.1, except the convexity property, and for which no quasi-equilibrium exists. This shows that the "convexifying effect of aggregation" is no longer valid for production economies with non-transitive preferences.

[8] A subset X of $\mathbb{L}$ is star-shaped about $x^0 \in X$ if for all $x \in X$ the line segment $[x^0, x]$ lie in X.

Counterexample 2.3.1. We consider the following private ownership economy, with two commodities and one producer

$$\mathcal{E} = ((T, \mathcal{L}(T), \lambda), \langle \mathbb{R}^2, \mathbb{R}^2 \rangle, (X, P, e), (Y, \theta)),$$

where the continuum T is the unit interval equipped with Lebesgue measure. The production set is $Y := -\mathbb{R}^2_+$. For each $a \in T$, the consumption set is $X(a) := \mathbb{R}^2_+$, the initial endowment is $e(a) := (1,1)$, the share is $\theta(a) = 1$ and the preferred sets are defined by

$$\forall x \in \mathbb{R}^2_+ \quad P_a(x) := P(x) = \{x' \in \mathbb{R}^2_+ \mid x'_1 > x_1 \quad \text{or} \quad x'_2 > x_2\}.$$

The economy $\mathcal{E}$ satisfies Assumptions M, P, S, B, LNS, FD and C_n without the convexity property. But $\mathcal{E}$ has no Walrasian quasi-equilibrium. Indeed, for each positive price $p \in \mathbb{L}_+ \setminus \{0\}$, we define the demand set

$$D(p) := \{x \in B(p) \mid P(x) \cap B(p) = \emptyset\},$$

where $B(p) := \{x \in \mathbb{R}^2_+ \mid p\,(x) \leqslant p\,((1,1))\}$ is the budget set. We then easily check that for all $p \in \mathbb{L}_+ \setminus \{0\}$, $D(p) = \emptyset$.

We provide hereafter two examples of production economies for which Theorem 2.3.1 applies but which are not covered by the existence results of Auman [4], Schmeidler [30] and Hildenbrand [21].

Example 2.3.1. We consider an economy with two goods, i.e., $\mathbb{L} = \mathbb{L}^* = \mathbb{R}^2$, one producer and the unit interval endowed with the Lebesgue measure $([0,1], \mathcal{L}[0,1], \lambda)$ as the measure set of agents. The production set correspondance Y is defined by

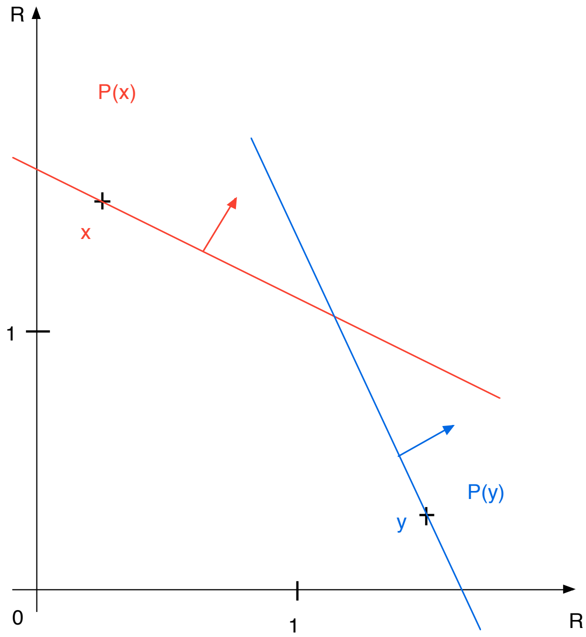
$$\forall a \in [0,1] \quad Y(a) := \{(y_1, y_2) \in \mathbb{R}^2 \mid \max(y_1, y_2) \leqslant 1\}.$$

For each agent $a \in [0,1]$, the intial endowment is $e(a) := (2-a, 2-a)$, the consumption set is

$$X(a) := \{(x_1, x_2) \in \mathbb{R}^2 \mid \min(x_1, x_2) \geqslant 0\},$$

and the preference correspondence P is defined by

$$\forall x = (x_1, x_2) \in X(a) \quad P_a(x) := \{x' \in X \mid \langle (1, a x_2), x' - x \rangle > 0\}.$$



The economy $\mathcal{E} = (([0,1], \mathcal{L}[0,1], \lambda), \langle \mathbb{R}^2, \mathbb{R}^2 \rangle, X, Y, P, e)$ satisfies the assumption of Theorem 2.3.1. But for each agent, the preference relation is not transitive, hence the existence of a Walrasian equilibrium for $\mathcal{E}$ is not covered by the existence results of Aumann [4], Schmeidler [30] and Hildenbrand [21].

Example 2.3.2. We consider an economy with two goods, i.e., $\mathbb{L} = \mathbb{L}^* = \mathbb{R}^2$, one producer and the unit interval endowed with the Lebesgue measure $([0,1], \mathcal{L}[0,1], \lambda)$ as the measure set of agents. The production set correspondence Y is defined by

$$\forall a \in [0,1] \quad Y(a) := \{(y_1, y_2) \in \mathbb{R}^2 \mid \max(y_1, y_2) \leqslant 1\}.$$

Let $a \in [0,1]$ be an agent. The initial endowment is $e(a) := (2-a, 2-a)$. For each $a \leqslant \lambda < 1$, we let

$$A_\lambda := [(\lambda, 0); (1,1)] \bigcup [(0,\lambda); (1,1)] \setminus \{(1,1)\}$$

and for each $1 \leqslant \lambda < +\infty$, we let

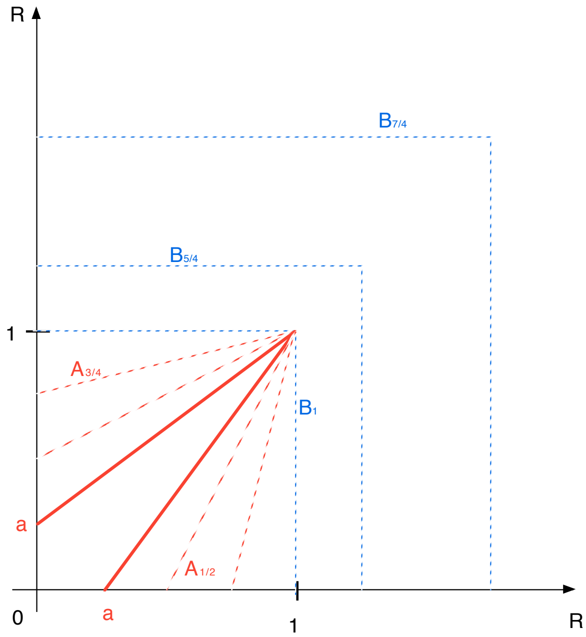
$$B_\lambda := [(\lambda, 0); (\lambda, \lambda)] \bigcup [(0, \lambda); (\lambda, \lambda)].$$

The consumption set of agent $a \in [0,1]$ is defined by

$$X(a) := \bigcup_{a \leqslant \lambda < 1} A_\lambda \cup \bigcup_{1 \leqslant \lambda < +\infty} B_\lambda.$$

Now we define the preference correspondence P_a as follows:

$$\forall x \in X(a) \quad P_a(x) := \begin{cases} \bigcup_{\lambda < \lambda' < 1} A_{\lambda'} \cup \bigcup_{1 \leqslant \lambda' < +\infty} B_{\lambda'} \setminus \{(1,1)\} & \text{if} \quad x \in A_\lambda \\ \bigcup_{\lambda < \lambda' < +\infty} B_{\lambda'} & \text{if} \quad x \in B_\lambda. \end{cases}$$



The economy $\mathcal{E} = (([0,1], \mathcal{L}[0,1], \lambda), \left\langle \mathbb{R}^2, \mathbb{R}^2 \right\rangle, X, Y, P, e)$ satisfies the assumption of Theorem 2.3.1. But for each agent, the preference relation is not negatively transitive, nor monotone, and the consumption sets are not convex, hence the existence of a Walrasian equilibrium for $\mathcal{E}$ is not covered by the existence results of Aumann [4], Schmeidler [30] and Hildenbrand [21].

2.3.5 Existence of equilibria for economies with monotone preferences

Assumption (MON). *[**Monotonicity**] For each agent $a \in A$, the consumption set $X(a)$ is convex comprehensive* [9] *and preferences are strictly monotone, that is, for each bundle $x \in X(a)$,*

$$\forall m > 0 \quad x + m \in \operatorname{co} P_a(x).$$

Remark 2.3.13. Usually in the literature, e.g. in Aumann [4], the consumption sets coincide with $\mathbb{L}_+$ and the strict-preferred sets are supposed to be strictly monotone, that is, for all $m > 0$, $x+m \in P_a(x)$.

Assumption (E). *[**Endowments**] There exists $(\bar{u}, \bar{v}) \in Y_\Sigma \times X_\Sigma$ such that $\omega + \bar{u} - \bar{v} \gg 0$.*

Remark 2.3.14. This assumption means that no commodity is totally absent from the market. In Aumann [4], it is supposed that $\omega \gg 0$. The Assumption E generalizes this assumption since in [4] the aggregate consumption set X_Σ coincide with $\mathbb{L}_+$ and the production sector is trivial.

In order to prove that a quasi-equilibrium of $\mathcal{E}$ is in fact an equilibrium, the economy will be required to satisfy the following assumption.

[9] That is $X(a) + \mathbb{L}_+ \subset X(a)$.

Assumption (WSS). *For almost every agent $a \in A$, one of the two following properties holds.*

(i) *There exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in \mathbb{L}_+$ and $X(a)$ is star-shaped at $x^0(a)$.*

(ii) $\{e(a)\} + Y(a) - X(a) \subset -\mathbb{L}_+$.

Remark 2.3.15. Survival Assumption S ensures that $0 \in \{e(a)\} + \tilde{Y}(a) - X(a)$. Assumption WSS(i) means that 0 is not the smallest non-negative vector in $\{e(a)\} + \tilde{Y}(a) - X(a)$. Assumption WSS will play the same role as Assumption SS introduced in the free-disposal on production framework, but SS is stronger than WSS. Indeed when preferences are strictly monotone, we prove the existence of a quasi-equilibrium with a price $p^* \gg 0$. This extra information allows us to lighten the Strong Survival Assumption SS.

Remark 2.3.16. In the framework of Aumann [4], the production sector is trivial, that is, for all $a \in A$, $Y(a) = 0$ and consumption sets coincide with the positive cone, that is, $X(a) = \mathbb{L}_+$. It follows that Assumption WSS is automatically valid. Indeed, Assumption S ensures that for almost every $a \in A$, $e(a) \in X(a) = \mathbb{L}_+$. If $e(a)$ is not zero, then WSS(i) is valid and if $e(a) = 0$, then it is WSS(ii) that is valid.

We present now, as a corollary of Theorem 2.3.1, a Walrasian equilibrium existence result for production economies with strictly monotone preferences.

Corollary 2.3.1. *If an economy satisfies Assumptions **C**, **M**, **P**, **S**, **B**, **MON** and **E**, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* \gg 0$. If moreover $\mathcal{E}$ satisfies **WSS** then (x^*, y^*, p^*) is a Walrasian equilibrium.*

Remark 2.3.17. This equilibrium existence result improves the Main Theorem in Aumann [4] and the Main Theorem in Schmeidler [30]. Indeed, Assumptions C, M, P, S, B, MON, E and WSS of Corollary 2.3.1 are implied by those used in [4] and [30]. Moreover Corollary 2.3.1 deals with production economies and not only with pure exchange economies, and it provides the existence of a Walrasian equilibrium without assuming that consumption sets coincide with the positive cone.

Proof. Following Remark 2.3.1, to prove Corollary 2.3.1, it is sufficient to prove the existence of a Walrasian quasi-equilibrium. Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S, B, MON and E. Once again we can suppose without any loss of generality that for almost every $a \in A$, $Y(a) = \tilde{Y}(a)$. We propose to construct an auxiliary economy $\mathcal{E}'$ *close* to $\mathcal{E}$ and satisfying Assumption FD, in order to apply Theorem 2.3.1. We let $\mathcal{E}' := ((A', \mathcal{A}', \mu'), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X', Y', P', e'))$ be the production economy with the measure space of agents $A' = A \cup \{\infty\}$, the σ-algebra $\mathcal{A}' = \mathcal{A} \cup \{B \cup \{\infty\} \mid B \in \mathcal{A}\}$, the measure μ' defined by $\mu'_{|\mathcal{A}} = \mu$, and for each $B \in \mathcal{A}$, $\mu'(B \cup \{\infty\}) = \mu(B) + 1$. The consumption sets correspondence X' is defined by $X'_{|A} = X$ and $X'(\infty) = \mathbb{L}_+$. The preference correspondence P' is defined by $P'_{|A} = P$ and $P'(\infty) := \{(x, y) \in \mathbb{L}^2_+ \mid y - x \in \operatorname{int} \mathbb{L}_+\}$ [10]. The production sets correspondence Y' is defined by $Y'_{|A} = Y$ and $Y'(\infty) = -\mathbb{L}_+$. The initial endowment function e' is defined by $e'_{|A} = e$ and $e'(\infty) = 0$. It is straightforward to verify that $\mathcal{E}'$ satisfies Assumptions C, M, P, S, B, FD and LNS. Applying Theorem 2.3.1, there exist an allocation $(x^*, y^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y)$, a price $p^* \in \mathbb{L}^*$ with $p^* \neq 0$ and bundles $(x^*(\infty), y^*(\infty)) \in \mathbb{L}_+ \times -\mathbb{L}_+$ satisfying the following properties.

(a) For almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a)) \ , \quad p^*(x^*(\infty)) = p^*(y^*(\infty))$$

and

$$x \in P_a(x^*(a)) \Rightarrow p^*(x) \geqslant p^*(x^*(a)) \ , \ x \in P'_\infty(x^*(\infty)) \Rightarrow p^*(x) \geqslant p^*(x^*(\infty)) \, .$$

(b) For almost every $a \in A$,

$$y \in Y(a) \Rightarrow p^*(y) \leqslant p^*(y^*(a)) \quad \text{and} \quad y \in Y'(\infty) \Rightarrow p^*(y) \leqslant p^*(y^*(\infty)) \, .$$

[10] Following Assumption E, the positive cone $\mathbb{L}_+$ has an interior point.

(c)

$$\int_A x^*(a)d\mu(a) + x^*(\infty) = \int_A e(a)d\mu(a) + \int_A y^*(a)d\mu(a) + y^*(\infty).$$

If we prove that $(x^*(\infty), y^*(\infty)) = (0,0)$ then (x^*, y^*, p^*) is a Walrasian quasi-equilibrium of $\mathcal{E}$. From (b) or (a), we have that $p^* \geqslant 0$ and applying Assumptions MON and E to (a), we check that $p^* \gg 0$. Since $0 \in Y'(\infty)$, applying (b) we check that $-\|y^*(\infty)\| \geqslant 0$. It follows that $y^*(\infty) = 0$ and applying (a), $\|x^*(\infty)\| \leqslant 0$. □

2.4 Discretization of measurable correspondences

2.4.1 Notations and definitions

We consider $(A, \mathcal{A}, \mu)$ a finite measure space and (D, d) a separable metric space. We recall that a function $f : A \to D$ is measurable if for each open subset $V \subset D$, $f^{-1}(V) := \{a \in A \mid f(a) \in V\} \in \mathcal{A}$, and a correspondence $F : A \twoheadrightarrow D$ is measurable if for each open subset $V \subset D$, $F^{-}(V) := \{a \in A \mid F(a) \cap V \neq \emptyset\} \in \mathcal{A}$.

Definition 2.4.1. A partition $\sigma = (A_i)_{i \in I}$ of A is a *measurable partition* if for all $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of A is *subordinated to the partition* σ if there exists a family $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $A^\sigma = \{a_i \mid i \in I\}$.

2.4.1.1 Simple functions subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the application which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i)\chi_{A_i},$$

where χ_{A_i} is the characteristic function [11] associated to A_i.

Definition 2.4.2. A function $s : A \to D$ is called *a simple function subordinated* to f if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $s = \phi(\sigma, A^\sigma)(f)$.

2.4.1.2 Simple correspondences subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the application which maps each measurable correspondence $F : A \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i \in I} F(a_i)\chi_{A_i}.$$

Note that the sum is well defined since there exists at most one non zero factor.

Definition 2.4.3. A correspondence $S : A \to D$ is called a *simple correspondence subordinated* to a correspondence F if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $S = \psi(\sigma, A^\sigma)(F)$.

Remark 2.4.1. If f is a function from A to D, let $\{f\}$ be the correspondence from A into D, defined for all $a \in A$ by $\{f\}(a) := \{f(a)\}$. We check that

$$\psi(\sigma, A^\sigma)(F) = \{\phi(\sigma, A^\sigma)(f)\}.$$

[11] That is, for all $a \in A$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

2.4.1.3 Hyperspace

Definition 2.4.4. The space of all non-empty subsets of D is noted $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wijsman topology on $\mathcal{P}^*(D)$, that is the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x,.))_{x\in D}$. If $V \subset D$ is a subset of D, we note $V^- = \{Z \subset D \mid Z \cap V \neq \emptyset\}$, and we note $\mathcal{E}(D)$ the Effrös σ-algebra, that is the σ-algebra generated by all sets V^-, where V is open.

Hess proved in [19] that, restricted to the set of non-empty closed subsets of D, the Effrös σ-algebra $\mathcal{E}(D)$ and the Borel σ-algebra $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$ relative to the Wijsman topology coincide. In fact this result is still true if we do not restrict to closed subsets.

Theorem 2.4.1 (Hess).

$$\mathcal{E}(D) = \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}).$$

Proof. If $x \in D$, $\alpha > 0$ and $Z \subset D$, then we note

$$B(x,\alpha) = \{z \in D \mid d(x,z) < \alpha\} \quad \text{and} \quad \delta_x(Z) := d(x,Z).$$

We easily check that

$$\delta_x^{-1}([0,\alpha[) = [B(x,\alpha)]^- .$$

It follows that (we do not make use of separability) $\mathcal{B}(\mathcal{P}^*(D), \tau_{W_d}) \subset \mathcal{E}(D)$. Since D is separable, each open set in D is a countable union of open balls. It follows that $\mathcal{E}(D) \subset \mathcal{B}(\mathcal{P}^*(D), \tau_{W_d})$. □

Remark 2.4.2. A direct corollary of Theorem 2.4.1 is that a correspondence F from A into D is measurable if and only if for all $x \in D$, the real valued function $d(x, F(.))$ is measurable.

Definition 2.4.5. The Hausdorff semi-metric H_d on $\mathcal{P}^*(D)$ is defined by

$$\forall (A,B) \in \mathcal{P}^*(D) \quad H_d(A,B) := \sup\{|d(x,A) - d(x,B)| \mid x \in D\}.$$

A subset C of D is the Hausdorff limit of a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of D, if

$$\lim_{n\to\infty} H_d(C_n, C) = 0.$$

2.4.2 Approximation of measurable real valued functions

We propose to prove that for a countable set of measurable real valued functions, there exists a sequence of measurable partitions *approximating* each function in the following sense.

Theorem 2.4.2. *Let $\mathcal{F}$ be a countable set of measurable real valued functions. There exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i\in I^n}$ of A, satisfying the following properties.*

(i) *Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $f \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple function $f^n := \phi(\sigma^n, A^n)(f)$ subordinated to f.*

1. *The function f is the pointwise limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*
2. *If $f(A)$ is bounded then f is the uniform limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*

(ii) *If $\mathcal{G} \subset \mathcal{F}$ is a finite subset of integrable functions, then there exists a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $n \in \mathbb{N}$,*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Proof. Let $f : A \to \mathbb{R}_+$ be a measurable function. We will construct a sequence of measurable partitions depending on f. Let $n \in \mathbb{N}$, we define $K^n = \{0, \dots, 2^{2n}\}$. We define the measurable partition $\pi^n(f) = (E^n_k(f))_{k \in K^n}$ by

$$E^n_k(f) = \begin{cases} f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right) & \text{if } k \in \{0, \dots, 2^{2n} - 1\}, \\ \\ f^{-1}([2^n, +\infty[) & \text{if } k = 2^{2n}. \end{cases}$$

Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ be a countable set of real valued measurable functions. Now for each $n \in \mathbb{N}$, we define $\mathcal{F}^n := \{f_k \mid 0 \leqslant k \leqslant n\}$ and σ^n as the following measurable partition

$$\sigma^n := (A^n_i)_{i \in I^n} \subset (A^n_i)_{i \in S^n} := \bigvee_{f \in \mathcal{F}^n} [\pi^n(f_+) \vee \pi^n(f_-)],$$

where $I^n := \{i \in S^n \mid A^n_i \neq \emptyset\}$ and $\vee$ is the natural supremum operator on partitions.

We begin to prove part (i) of Theorem 2.4.2. Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n, let $f \in \mathcal{F}$ and $a \in A$. Following the construction of σ^n, we can suppose, without any loss of generality, that $f = f_+$. For all n large enough, $f \in \mathcal{F}^n$ and $f(a) < 2^n$, and following the construction of the partition σ^n, for all n large enough

$$\forall b \in A^n_i \quad |f(b) - f(a)| \leqslant \frac{1}{2^n},$$

where $i \in I^n$ is such that $a \in A^n_i$. It follows that $\lim_{n \to \infty} f^n(a) = f(a)$, and this limit is uniform if $f(A)$ is bounded.

We now prove part (ii) of Theorem 2.4.2. Let $\mathcal{G} \subset \mathcal{F}$, be a finite set of integrable functions. Once again, we can suppose that all functions in $\mathcal{G}$ are positive. We let $h := \sum_{f \in \mathcal{G}} f$, this function defined from A to $\mathbb{R}_+$ is integrable. For each $n \in \mathbb{N}$, for each $i \in I^n$, A^n_i is non-empty and we can choose $a^n_i \in A^n_i$ such that

$$h(a^n_i) \leqslant 1 + \inf\{h(b) \mid b \in A^n_i\}.$$

We have constructed a sequence $(A^n)_{n \in \mathbb{N}}$ of finite sets $A^n := \{a^n_i \mid i \in I^n\}$, subordinated to the measurable partition σ^n, such that for each $f \in \mathcal{G}$, for each $n \in \mathbb{N}$,

$$\forall a \in A \quad f^n(a) \leqslant 1 + h(a).$$

Applying part (i) and the Lebesgue Dominated Convergence Theorem,

$$\forall f \in \mathcal{G} \quad \lim_{n \to \infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

□

2.4.3 Approximation of measurable correspondences

As a corollary of Theorem 2.4.2, we propose to prove that for a countable set of measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence in the following sense.

Corollary 2.4.1. *Let $\mathcal{F}$ be a countable set of measurable correspondences with non-empty values from A into D and let $\mathcal{G}$ be a finite set of integrable functions from A to $\mathbb{R}$. There exists a sequence $(\sigma^n)_{n \in \mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A^n_i)_{i \in I^n}$ of A, satisfying the following properties.*

(a) *Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $F \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F. The following properties are then satisfied.*

1. *For all $a \in A$, $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_{n\in\mathbb{N}}$, i.e. ,*

$$\forall a \in A \quad \forall x \in A \qquad \lim_{n\to\infty} d(x, F^n(a)) = d(x, F(a)).$$

2. *If D is d-bounded then for all $x \in D$ the real valued function $d(x, F(.))$ is the uniform limit of the sequence $(d(x, F^n(.)))_{n\in\mathbb{N}}$.*
3. *If D is d-totally bounded [12] then F is the uniform Hausdorff limit of the sequence $(F^n)_{n\in\mathbb{N}}$.*

(b) *There exists a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $n \in \mathbb{N}$, if we let $f^n := \phi(\sigma^n, A^n)(f)$ be the simple function subordinated to each $f \in \mathcal{G}$, then*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remark 2.4.3. The property (a1) implies in particular that, if $(x^n)_{n\in\mathbb{N}}$ is a sequence of D, d-converging to $x \in D$, then

$$\forall a \in A \qquad \lim_{n\to\infty} d(x^n, F^n(a)) = d(x, F(a)).$$

It follows that if F is non-empty closed valued, then property (a1) implies that

$$\forall a \in A \quad \mathrm{ls}\, F^n(a) \subset F(a).$$

Proof. If $F : A \twoheadrightarrow D$ is a correspondence, we consider the distance function associated to F, $\delta_F : A\times D \to \mathbb{R}_+$ defined by $\delta_F : (a, x) \mapsto \mathrm{d}(x, F(a))$. Let $F \in \mathcal{F}$, following Theorem 2.4.1, F is measurable if and only if, for all $x \in D$, $\delta_F(., x)$ is measurable. Since D is separable, there exist a sequence $(x_n)_{n\in\mathbb{N}}$ dense in D. We let, for each $n \in \mathbb{N}$, $\delta_n^F := \delta_F(., x_n)$. If $f \in \mathcal{G}$, we let $|f(.)| : A \to \mathbb{R}_+$ defined by $a \mapsto |f(a)|$. We define

$$\mathcal{F}_0 = \{|f(.)| \mid f \in \mathcal{G}\} \cup \bigcup_{F\in\mathcal{F}} \{\delta_n^F \mid n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{G}_0 = \{|f(.)| \mid f \in \mathcal{G}\}.$$

Note that, if F is a correspondence from A into D, then for all measurable partition σ of A, and for each subset A^σ subordinated to σ,

$$\forall x \in L \quad \phi(\sigma, A^\sigma)(d(x, F(.)) = d(x, \psi(\sigma, A^\sigma)(F)(.)).$$

We then apply Theorem 2.4.2 to the countable set $\mathcal{F}_0$ of measurable functions and the finite set $\mathcal{G}_0$ of integrable functions. Noting that, for each $a \in A$, for all $F \in \mathcal{F}$, the functions $\delta_F(a, .)$ are 1-Lipschitz, we easily get the desired result. ☐

As a corollary of Corollary 2.4.1, we propose to prove that for a countable set of measurable functions, there exists a sequence of measurable partitions *approximating* each function in the following sense.

Corollary 2.4.2. *Let $\mathcal{F}$ be a countable set of measurable functions from A to D and let $\mathcal{G}$ be a finite set of integrable functions from A to $\mathbb{R}$. There exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i\in I^n}$ of A, satisfying the following properties.*

(a) *Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $f \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple function $f^n := \phi(\sigma^n, A^n)(f)$ subordinated to f. The following properties are then satisfied.*

[12] That is for each $\varepsilon > 0$ there exists a finite subset $\{x_1, \cdots, x_n\} \subset D$ such that the collection of balls $B(x_i, \varepsilon) = \{z \in D \mid d(z, x_i) < \varepsilon\}$ covers D.

1. *The function f is the pointwise d-limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*
2. *If D is d-totally bounded then f is the d-uniform limit of the sequence $(f^n)_{n\in\mathbb{N}}$.*

(b) *There exists a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $n \in \mathbb{N}$,*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remark 2.4.4. This result generalizes Theorem 4.38 in Aliprantis and Border [1].

Proof. For each function f from A to D, consider the correspondence F from A into D, defined by

$$\forall a \in A \quad F(a) := \{f(a)\},$$

and apply Corollary 2.4.1. □

2.5 Proof of the main existence result

2.5.1 Stronger existence results

We will prove in fact stronger existence results than Theorem 2.3.1 and Corollary 2.3.1. Hereafter we present the Assumptions C'$_n$, C'$_p$, M' and SS' which are weaker than, respectively Assumptions C$_n$, C$_p$, M and SS.

Assumption (C'$_n$). *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is lower semi-continuous* [13]*,*

(ii) *the preference relation $P(a)$ is convex* [14]*.*

Remark 2.5.1. The properties required in Assumption C'$_n$, are the natural extension of those required in the finite agent's set-up to prove the existence of a quasi-equilibrium.

Assumption (C'$_p$). *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed and P_a is lower semi-continuous,*

(ii) *if a belongs to the non-atomic part of $(A, \mathcal{A}, \mu)$ then $P(a) \subset \tilde{P}(a)$ where $\tilde{P}(a)$ is an ordered binary relation on $X(a)$ with open lower sections* [15] *in $X(a)$ and if a belongs to an atom of $(A, \mathcal{A}, \mu)$ then the preference relation $P(a)$ is convex.*

Remark 2.5.2. Let $a \in A$, following Sondermann [32], if $P(a)$ is partially ordered and continuous [16] then there exists an upper semi-continuous function $u_a : X(a) \to \mathbb{R}$ such that $P(a) \subset \{(x, y) \in X(a) \times X(a) \mid u_a(x) < u_a(y)\}$. The function u_a defines an ordered binary relation $\widetilde{P}(a)$ on $X(a)$ with open lower sections such that $P(a) \subset \widetilde{P}(a)$. It follows that in the frameworks of Aumann [4], Hildenbrand [21], Schmeidler [30] and Cornet and Topuzu [11], Assumption C'$_p$ is valid.

[13] That is for all open set $V \subset \mathbb{L}$, $\{x \in X(a) \mid P_a(x) \cap V \neq \emptyset\}$ is open in X(a).
[14] We recall that P_a is convex if $X(a)$ is convex and for all $x \in X(a)$, $x \notin \operatorname{co} P_a(x)$.
[15] That is for all $y \in X(a)$, $\{x \in X(a) \mid (x, y) \in \tilde{P}(a)\}$ is open in $X(a)$.
[16] That is for all $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$.

Remark 2.5.3. In general, Assumptions C'$_n$ and C'$_p$ are not comparable but if preferences are convex then Assumption C'$_p$ implies Assumption C'$_n$.

Assumption (C'). *Assumption C'$_p$ or Assumption C'$_n$ is satisfied.*

Assumption (M'). *The correspondences X and Y are graph measurable, that is*

$$\{(a,x) \in A \times \mathbb{L} \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}) \quad \textit{and} \quad \{(a,y) \in A \times \mathbb{L} \mid y \in Y(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L})$$

and the correspondence of preferences P is lower semi-graph measurable, that is, for each graph measurable correspondence $V : A \twoheadrightarrow \mathbb{L}$ with open values,

$$\{(a,x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}).$$

Remark 2.5.4. In the framework of Hildenbrand [21] and Cornet and Topuzu [11], the correspondence P is supposed to be graph measurable. Since $(A, \mathcal{A}, \mu)$ is complete, applying Proposition 2.7.6, it follows that P is lower semi-graph measurable and Assumption M' is then valid.

Remark 2.5.5. In the framework of Aumann [4] and Schmeidler [30], it is assumed that preferences are "Aumann measurable". Applying Proposition 2.7.5, the preferences P are then graph measurable and Assumption M' is valid.

Remark 2.5.6. Under Assumption C', the correspondence X is closed valued and for each graph measurable correspondence $V : A \twoheadrightarrow \mathbb{L}$ with open values, the correspondence $R_V : a \mapsto \{(a,x) \in G_X \mid P_a(x) \cap V(a) = \emptyset\}$ is closed valued. If we suppose that Y is closed valued, then following Proposition 2.7.2, Assumption M' is valid if and only if the correspondences X and Y are measurable and for all graph measurable correspondence $V : A \twoheadrightarrow \mathbb{L}$ with open values, the correspondence R_V is measurable. It follows that if A is a finite set and $\mathcal{A} = 2^A$, Assumption M' is then automatically valid.

Assumption (SS'). *For almost every agent $a \in A$, there exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in int\mathbb{L}_+$, $X(a)$ is star-shaped about $x^0(a)$ and for all $x \in X(a)$, $P_a(x)$ is radial to $x^0(a)$* [17].

Remark 2.5.7. If $X(a)$ is star-shaped about $x^0(a)$ and $P_a(x)$ is open in $X(a)$ then $P_a(x)$ is radial to $x^0(a)$.

Theorem 2.5.1. *If an economy $\mathcal{E}$ satisfies Assumptions C', M', P, S, B, LNS and FD, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exits, with $p^* > 0$. If moreover $\mathcal{E}$ satisfies SS' then (x^*, y^*, p^*) is a Walrasian equilibrium of $\mathcal{E}$.*

Assumption (WSS'). *For almost every agent $a \in A$, one of the two following properties holds.*

(i) *There exists $x^0(a) \in X(a)$ and $y^0(a) \in \tilde{Y}(a)$ such that $e(a) + y^0(a) - x^0(a) \in \mathbb{L}_+$, $X(a)$ is star-shaped about $x^0(a)$ and for all $x \in X(a)$, $P_a(x)$ is radial to $x^0(a)$.*

(ii) $\{e(a)\} + Y(a) - X(a) \subset -\mathbb{L}_+$.

Corollary 2.5.1. *If an economy satisfies Assumptions C', M', P, S, B, MON and E, then a Walrasian quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* \gg 0$. If moreover $\mathcal{E}$ satisfies WSS' then (x^*, y^*, p^*) is a Walrasian equilibrium.*

2.5.2 Satiation equilibria

Hereafter, we introduce an auxiliary concept of quasi-equilibria for an economy $\mathcal{E}$.

Definition 2.5.1. An element (x^*, y^*, p^*) of $\mathrm{S}^1(X) \times \mathrm{S}^1(Y) \times \mathbb{L}^*$ is a *satiation quasi-equilibrium* of the economy $\mathcal{E}$ if $p^* \neq 0$ and if the following properties are satisfied.

[17] A subset P of $\mathbb{L}$ is radial to $x^0 \in X$ if for each $y \in P$ the segment $[y, y + \lambda(x_0 - y)]$ still lies in P for some $0 < \lambda \leqslant 1$.

(i) For almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*(x) \geqslant p^*(y) + p^*(e(a)).$$

(ii)

$$\int_A x^*(a) d\mu(a) = \int_A e(a) d\mu(a) + \int_A y^*(a) d\mu(a).$$

Remark 2.5.8. When the condition (i) is replaced by the following condition

(i')

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*(x) > p^*(y) + p^*(e(a)),$$

then (x^*, y^*, p^*) is called a satiation equilibrium. Indeed, condition (i') means that either agent $a \in A$ is satiated, $P_a(x^*(a)) = \emptyset$ or for all bundle $x \in X(a)$, if x is prefered to $x^*(a)$ then x is not in the bugdet set, $p^*(x) > p^*(e(a)) + \sup p^*(Y(a))$. Note that the consumption bundle $x^*(a)$ is not expected to lie in the budget set, however the consumption plan x^* has to be realizable.

If (x^*, y^*, p^*) is a Walrasian quasi-equilibrium of an economy $\mathcal{E}$, then (x^*, y^*, p^*) is clearly a satiation quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, a suitable *Local Non Satiation* property on $\mathcal{E}$ under which the converse is true.

Remark 2.5.9. A satiation quasi-equilibrium (x^*, y^*, p^*) of an economy $\mathcal{E}$, is a Walrasian quasi-equilibrium, if we assume that, for almost every agent $a \in A$, for all bundle $x \in X(a)$, $x \in \overline{\text{co}}\, P_a(x)$.

Following this remark, to prove Theorem 2.3.1, it is sufficient to prove the following lemma.

Lemma 2.5.1. *If $\mathcal{E}$ is an economy satisfying Assumptions C', M', P, S, B and FD then a satiation quasi-equilibrium (x^*, y^*, p^*) exists, with $p^* > 0$.*

2.5.3 Existence of satiation equilibria for integrably bounded economies

As an auxiliary result, we propose to first prove existence of a satiation equilibrium for integrably bounded economies, that is economies satisfying the following assumption.

Assumption (IB). *The consumption sets correspondence X and the production sets correspondence Y are integrably bounded* [18].

This first step allows us to isolate the crucial aspect of the new approach, which is the approximation of economies with a measure space of agents (measurable correspondences) by a sequence of economies with a finite set of agents (resp. simple correspondences). Moreover, the framework of integrably bounded economies allows us to deal with both non-ordered but convex preferences and partially ordered but non-convex preferences. This auxiliary result will be applied in the next subsection to prove Lemma 2.5.1.

Lemma 2.5.2. *If $\mathcal{E}$ is an economy satisfying Assumptions C', M', P, S and IB, then a satiation quasi-equilibrium exists.*

Proof. Following Proposition 2.7.7, we can suppose without any loss of generality that for almost every $a \in A$, $Y(a) = \tilde{Y}(a)$ and $e(a) = 0$. Following Remark 2.5.6, the correspondences X, Y are measurable and following Proposition 2.7.1, there exist a sequence $(f_k)_{k \in \mathbb{N}}$ of measurable selections of X and a sequence $(g_k)_{k \in \mathbb{N}}$ of measurable selections of Y such that for all $a \in A$,

$$X(a) = \text{cl}\,\{f_k(a) \mid k \in \mathbb{N}\} \quad \text{and} \quad Y(a) = \text{cl}\,\{g_k(a) \mid k \in \mathbb{N}\}.$$

We let for all $(k, q) \in \mathbb{N}^2$, $R_{k,q}(a) := \{x \in X(a) \mid P_a(x) \cap B(f_k(a), r_q) = \emptyset\}$, where $r_q = 1/(q + 1)$ and $B(f_k(a), r_q)$ is the open ball centered in $f_k(a)$ and of radius r_q. For all $(k, q) \in \mathbb{N}^2$, $R_{k,q}$ is graph measurable with closed values, following Proposition 2.7.1 it is then measurable.

[18] That is, there exists an integrable function $h : A \to \mathbb{R}_+$ such that for almost every $a \in A$, for all $(x, y) \in X(a) \times Y(a)$, $\max\{\|x\|, \|y\|\} \leqslant h(a)$.

Note that for almost every agent $a \in A$, for all $x \in \mathbb{L}$,

$$[d(x, X(a)) = 0 \Leftrightarrow x \in X(a)] \quad \text{and} \quad [d(x, Y(a)) = 0 \Leftrightarrow x \in Y(a)],$$

and if $x \in X(a)$,

$$d(x, R_{k,q}(a)) > 0 \Longleftrightarrow P_a(x) \cap B(f_k(a), r_q) \neq \emptyset.$$

Following Assumption IB, there exists an integrable function $h : A \to \mathbb{R}_+$ such that for almost every $a \in A$, for all $(x, y) \in X(a) \times Y(a)$, $\max\{\|x\|, \|y\|\} \leqslant h(a)$. Applying Corollary 2.4.1, there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of measurable partitions $\sigma^n = (A_i^n)_{i\in S^n}$ of $(A, \mathcal{A})$, and a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets $A^n = \{a_i^n \mid i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying the following properties.

Fact 2.5.1. For all $a \in A$,

(i) for all $n \in \mathbb{N}$,

$$h^n(a) \leqslant 1 + h(a) \quad \text{and} \quad \forall k \in \mathbb{N} \quad \lim_{n\to\infty} (f_k^n(a), g_k^n(a)) = (f_k(a), g_k(a)) \ ;$$

(ii) for all sequence $(x^n)_{n\in\mathbb{N}}$ of $\mathbb{L}$ converging to $x \in \mathbb{L}$,

$$\lim_{n\to\infty} d(x^n, X^n(a)) = d(x, X(a)) \ , \quad \lim_{n\to\infty} d(x^n, Y^n(a)) = d(x, Y(a))$$

and

$$\forall (k, q) \in \mathbb{N}^2 \quad \lim_{n\to\infty} d(x^n, R_{k,q}^n(a)) = d(x, R_{k,q}(a)).$$

We construct now a sequence of economies with a finite set of consumers. We distinguish two cases. In the first case (Claim 2.5.1) preferences are possibly non-ordered but convex, in the second case (Claim 2.5.2) preferences are ordered but possibly non-convex.

Claim 2.5.1. If $\mathcal{E}$ satisfies C_n, then a satiation quasi-equilibrium exists.

Proof. For all $n \in \mathbb{N}$, we note $\mathcal{E}^n$ the following *finite* economy

$$\mathcal{E}^n = (\langle \mathbb{L}^*, \mathbb{L}\rangle, (X_i^n, Y_i^n, P_i^n)_{i\in I^n}) \ ,$$

where $I^n := \{i \in S^n \mid \mu(A_i^n) \neq 0\}$ is the finite set of consumers. The consumption set of consumer $i \in I^n$ is given [19] by $X_i^n := \mu(A_i^n) X(a_i^n)$ and the production set is given by $Y_i^n := \mu(A_i^n) Y(a_i^n)$. Preferences are defined by the relation $P_i^n := \mu(A_i^n) P(a_i^n)$. For all $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem 2.6.1. It follows that, for all $n \in \mathbb{N}$, there exists

$$((x_i^n)_{i\in I^n}, (y_i^n)_{i\in I^n}, p^n) \in \prod_{i\in I^n} X_i^n \times \prod_{i\in I^n} Y_i^n \times \mathbb{L}^* \ ,$$

satisfying $\|p^n\| = 1$, $\sum_{i\in I^n} x_i^n = \sum_{i\in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n(x - y) \geqslant 0$. Let, for all $n \in \mathbb{N}$,

$$x^n := \sum_{i\in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i\in I^n} \frac{y_i^n}{\mu(A_i^n)} \chi_{A_i^n}.$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying [20]

$$\int_A x^n(a) d\mu(a) = \int_A y^n(a) d\mu(a). \tag{2.1}$$

$$\forall a \in \bigcup_{i\in I^n} A_i^n \quad (x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n(x) \geqslant p^n(y). \tag{2.2}$$

[19] The consumer a_i^n *represents* the coalition A_i^n.

[20] Following the notations of Section 2.7, $P^n := \psi(\sigma^n, A^n)(P)$, that is, for all $a \in A$, $P^n(a) = P(a_i^n)$, where $i \in I^n$ is such that $a \in A_i^n$.

Following (i) of Fact 2.5.1, the sequences $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$ are integrably bounded. Applying Theorem 2.7.1 there exist integrable functions $x^*, y^* : A \to \mathbb{L}$, such that

$$\int_A (x^*, y^*) = \lim_{n\to\infty} \int_A (x^n, y^n)$$

and

$$\text{for a.e. } a \in A \quad (x^*(a), y^*(a)) \in \operatorname{ls}\{(x^n(a), y^n(a))\}.$$

Since, for all $n \in \mathbb{N}$, $\|p^n\| = 1$, there exists a subsequence of $(p^n)_{n\in\mathbb{N}}$ converging to p^*, with $\|p^*\| = 1$.

We propose to prove that (x^*, p^*) is a satiation quasi-equilibrium of $\mathcal{E}$. We let

$$A_0 := \bigcup_{n\in\mathbb{N}} \bigcup_{i\in S^n\setminus I^n} A_i^n\,,$$

then we easily check that $\mu(A_0) = 0$. Let now A' be a subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost every where* assumptions and properties are satisfied for all $a \in A'$.

To prove condition (ii) of Definition 2.5.1, it is sufficient to prove that $(x^*, y^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y)$. First let us prove that, for all $a \in A'$, $x^*(a) \in X(a)$. Let $a \in A'$, by construction, we have that for every $n \in \mathbb{N}$, $x^n(a) \in X^n(a)$, and thus, for every $n \in \mathbb{N}$, $d(x^n(a), X^n(a)) = 0$. Since $x^*(a) \in \operatorname{ls}\{x^n(a)\}$, we apply Fact 2.5.1 to get that $d(x^*(a), X(a)) = 0$. We prove similarly that $y^* \in \mathrm{S}^1(Y)$. In fact we proved that for almost every $a \in A$,

$$\operatorname{ls}(X^n(a)) \subset X(a) \quad \text{and} \quad \operatorname{ls}(Y^n(a)) \subset Y(a).$$

We will now prove that (x^*, p^*) satisfies condition (i) of Definition 2.5.1. Let $a \in A'$ and $(x, y) \in P_a(x^*(a)) \times Y(a)$. Since $Y(a) = \operatorname{cl}\{g_k(a) \mid k \in \mathbb{N}\}$, there exist a subsequence $(g_{\psi(k)}(a))_{k\in\mathbb{N}}$ converging to y. To prove that $p^*(x) \geqslant p^*(y)$, it is sufficient to prove that for all k and q large enough, there exists [21] $z \in \overline{B}(x, 2r_q)$ such that $p^*(z) \geqslant p^*\left(g_{\psi(k)}(a)\right)$. Let $j \in \psi(\mathbb{N})$ and $q \in \mathbb{N}$. Since $X(a) = \operatorname{cl}\{f_k(a) \mid k \in \mathbb{N}\}$ there exists $k \in \mathbb{N}$ such that $f_k(a) \in B(x, r_q)$. In particular $x \in B(f_k(a), r_q) \cap P_a(x^*(a))$ and $d(x^*(a), R_{k,q}(a)) > 0$. Applying Fact 2.5.1, for all n large enough, $d(x^n(a), R^n_{k,q}(a)) > 0$. It follows that there exists $z^n \in P^n_a(x^n(a)) \cap B(f^n_k(a), r_q)$. Thus, applying (2.2), for all n large enough, $p^*(z^n) \geqslant p^n\left(g^n_j(a)\right)$. Now the sequence $(f^n_k(a))_{n\in\mathbb{N}}$ converges to $f_k(a)$, thus $(z^n)_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, $(z^n)_{n\in\mathbb{N}}$ converges to $z \in \mathbb{L}$ which satisfies $p^*(z) \geqslant p^*(g_j(a))$ and $d(z, f_k(a)) \leqslant r_q$, that is, $z \in \overline{B}(x, 2r_q)$. □

We consider now the case of ordered but possibly non-convex preferences.

Claim 2.5.2. If $\mathcal{E}$ satisfies C_p, then a satiation quasi-equilibrium exists.

Proof. The purely atomic part of A is noted A^{pa} and the non-atomic part of A is noted A^{na}. Under Assumption C'$_p$, for almost every $a \in A^{na}$, there exists an ordered binary relation $\tilde{P}(a)$ on $X(a)$ such that $P(a) \subset \tilde{P}(a)$. We let, for every $a \in A^{pa}$, $\tilde{P}(a) := P(a)$. We define the correspondence $\tilde{R}$ from A into $\mathbb{L} \times \mathbb{L}$ by, for each $a \in A$, $\tilde{R}(a) := \{(z, z') \in X(a) \times X(a) \mid (z', z) \notin \tilde{P}(a)\}$.

In order to use the same *limit* argument as Claim 2.5.1, we define preferences satisfying the *convexity property*. This construction is borrowed from Hildenbrand [22]. We let, for each $a \in A$, $\hat{X}(a) := \operatorname{co} X(a)$ and we define $\hat{P} : A \to \mathbb{L} \times \mathbb{L}$ by, for almost every $a \in A$,

$$\hat{P}(a) := \{(x, x') \in \hat{X}(a) \times \hat{X}(a) \mid x' \in X(a) \quad \text{and} \quad x \notin \operatorname{co} \tilde{R}_a(x')\}.$$

Note that for all $a \in A^{pa}$, $\hat{X}(a) = X(a)$ and $\hat{P}(a) = P(a)$. For almost every $a \in A^{na}$, the preferences $\tilde{P}(a)$ have open lower sections, it follows that for almost every $a \in A^{na}$, for each $y \in \hat{X}(a)$, $\hat{P}_a^{-1}(y)$ is open in $\hat{X}(a)$. Moreover, the binary relation $\tilde{R}(a)$ is a complete pre-order on $X(a)$. We check then, that for almost every $a \in A$, $\hat{P}(a)$ satisfies the following convexity property,

$$\forall x \in \hat{X}(a) \quad x \notin \operatorname{co} \hat{P}_a(x).$$

[21] For each $y \in \mathbb{L}$ and $r > 0$, we define $\overline{B}(y, r) = \{z \in \mathbb{L} \mid d(z, y) \leqslant r\}$.

We are now ready to construct the sequence of finite-consumers economies. For all $n \in \mathbb{N}$, we note $\mathcal{E}^n$ the following *finite* economy $\mathcal{E}^n = (\langle \mathbb{L}^*, \mathbb{L}\rangle, (X_i^n, Y_i^n, P_i^n)_{i \in I^n})$ where $I^n := \{i \in S^n \mid \mu(A_i^n) \neq 0\}$ is the finite set of consumers. The consumption set of the consumer $i \in I^n$ is given by $X_i^n := \mu(A_i^n)\hat{X}(a_i^n)$ and the production set is given by $Y_i^n := \mu(A_i^n)[Y(a_i^n) + (1/n)\overline{B}]$, where $\overline{B}$ is the closed unit ball in $\mathbb{L}$. Preferences are defined by the binary relation $P_i^n := \mu(A_i^n)\hat{P}(a_i^n)$. For all $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem 2.6.1. It follows that, for all $n \in \mathbb{N}$, there exists

$$((x_i^n)_{i \in I^n}, (y_i^n)_{i \in I^n}, p^n) \in \prod_{i \in I^n} X_i^n \times \prod_{i \in I^n} Y_i^n \times \mathbb{L}^*,$$

satisfying $\|p^n\| = 1$, $\sum_{i \in I^n} x_i^n = \sum_{i \in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n(x - y) \geqslant 0$. For all $n \in \mathbb{N}$, for all $i \in I^n$, there exists $\xi_i^n \in \overline{B}$ such that $y_i^n - (\mu(A_i^n)/n)\xi_i^n \in \mu(A_i^n)Y(a_i^n)$. For all $n \in \mathbb{N}$, we let $\xi^n := \sum_{i \in I^n} \mu(A_i^n)\xi_i^n \in \mu(A)\overline{B}$ and

$$x^n := \sum_{i \in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i \in I^n} \left(\frac{y_i^n}{\mu(A_i^n)} - \frac{1}{n}\xi_i^n \right) \chi_{A_i^n}.$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a) d\mu(a) = \int_A y^n(a) d\mu(a) + (1/n)\xi^n \tag{2.3}$$

$$\forall a \in \bigcup_{i \in I^n} A_i^n \quad (x, z) \in P_a^n(x^n(a)) \times (Y^n(a) + (1/n)\overline{B}) \Rightarrow p^n(x) \geqslant p^n(z). \tag{2.4}$$

Since, for all $n \in \mathbb{N}$, $\|p^n\| = 1$, there exists a subsequence of $(p^n)_{n \in \mathbb{N}}$ converging to p^*, with $\|p^*\| = 1$. For all $a \in A$, we let

$$B(a) = \{x \in X(a) \mid p^*(x) \leqslant \sup p^*(Y(a))\}$$

and

$$\beta(a) = \{x \in X(a) \mid p^*(x) < \sup p^*(Y(a))\}.$$

We define the correspondences D, G and H by, for all $a \in A$,

$$D(a) := \{x \in B(a) \mid P_a(x) \cap B(a) = \emptyset\}, \quad G(a) := \{x \in X(a) \mid P_a(x) \cap B(a) = \emptyset\}$$

and

$$H(a) := \{x \in X(a) \mid P_a(x) \cap \beta(a) = \emptyset\}.$$

When replacing X by $\hat{X}$ and P by $\hat{P}$, we define $\hat{G}$. Moreover, for each $n \in \mathbb{N}$, when replacing X by X^n, P by P^n, Y by Y^n and p^* by p^n, we define $B^n(a)$, $\beta^n(a)$, $D^n(a)$, $G^n(a)$. Similarly when replacing P^n by $\tilde{P}^n$, we define $\tilde{D}^n$ and $\tilde{G}^n$. We define $\hat{G}^n$ when X^n is replaced by $\hat{X}^n$ and P^n by $\hat{P}^n$. For all $n \in \mathbb{N}$, for all $a \in A^{pa}$, $\hat{G}^n(a) = \tilde{G}^n(a) = G^n(a)$. We assert that for all $n \in \mathbb{N}$,

$$\forall a \in A^{na} \quad \hat{G}^n(a) \subset \operatorname{co}[\tilde{G}^n(a)] \subset \operatorname{co}[G^n(a)]. \tag{2.5}$$

and Indeed, if $a \in A^{pa}$ then $\hat{P}^n(a) = P^n(a)$ and the result follows. Now let $a \in A^{na}$ and $x \in \hat{G}^n(a)$. The set $X^n(a)$ is compact, the strict-preference relation $\tilde{P}^n(a)$ is irreflexive, transitive with open lower sections. Hence, following a classical maximal argument, the set $\tilde{D}^n(a)$ is non-empty. Let $\tilde{x} \in \tilde{D}^n(a)$, then $\tilde{x} \in B^n(a)$, and since $x \in \hat{G}^n(a)$, we have that $(x, \tilde{x}) \notin \hat{P}^n(a)$, that is, $x \in \operatorname{co}\tilde{R}_a^n(\tilde{x})$. Since $\tilde{R}^n(a)$ is transitive and complete and $\tilde{x} \in \tilde{D}^n(a)$, it is straightforward to verify that $\tilde{R}_a^n(\tilde{x}) \subset \tilde{G}^n(a) \subset G^n(a)$, and thus $x \in \operatorname{co}[G^n(a)]$.

Since (x^n, p^n) satisfies (2.4), it follows [22] that for almost every $a \in A$, $x^n(a) \in \hat{G}^n(a) \subset \operatorname{co} G^n(a)$. Following (i) of Fact 2.5.1, the sequences $(x^n)_{n \in \mathbb{N}}$ and $(y^n)_{n \in \mathbb{N}}$ are integrably bounded. Applying Theorem 2.7.1, there exist integrable functions $x^*, y^* : A \to \mathbb{L}$, such that

$$\int_A (x^*, y^*) = \lim_{n \to \infty} \int_A (x^n, y^n)$$

[22] This is the reason why we introduce the unit ball $\overline{B}$ in the definition of Y_i^n.

and

$$\text{for a.e. } a \in A \quad (x^*(a), y^*(a)) \in \text{ls}\,\{(x^n(a), y^n(a))\}.$$

Following the arguments of Claim 2.5.1 almost verbatim, we prove that $(x^*, y^*) \in \mathrm{S}^1(X) \times \mathrm{S}^1(Y)$. Moreover, with (2.3) we get that

$$\int_A x^*(a) d\mu(a) = \int_A y^*(a) d\mu(a).$$

Once again, following the arguments of Claim 2.5.2 verbatim , we prove that for almost every $a \in A$,

$$\text{ls}\,(H^n(a)) \subset H(a).$$

Applying the Carathéodory Convexity Theorem, for almost every $a \in A$,

$$\text{ls}\;(\text{co}\,(H^n(a))) \subset \text{co}\,\text{ls}\,(H^n(a)) \subset \text{co}\,H(a).$$

It follows [23] that for almost every $a \in A$,

$$a \in A^{na} \Rightarrow x^*(a) \in \text{co}\,H(a) \quad \text{and} \quad a \in A^{pa} \Rightarrow x^*(a) \in H(a).$$

The correspondence β is graph measurable with open values (in $X(a)$), it follows from Assumption M' that the correspondence H is graph measurable. We apply now the Lyapunov Theorem,

$$\int_A y^* \in \int_{A^{na}} \text{co}\,[H(a)] d\mu(a) + \int_{A^{pa}} H(a) d\mu(a) = \int_A H(a) d\mu(a).$$

That is, there exists $\bar{x} \in \mathrm{S}^1(X)$ such that for almost every agent $a \in A$, $\bar{x}(a) \in H(a)$ and $\int_A \bar{x} \in Y_\Sigma$. It follows that $(\bar{x}, p^*)$ is a satiation quasi-equilibrium of the economy $\mathcal{E}$. □

The end of the proof of Lemma 2.5.2 is a direct consequence of Claims 2.5.1 and 2.5.2. □

2.5.4 Proof of Lemma 2.5.1

Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S, B and FD. In order to apply Lemma 2.5.2, we are led to truncate economies such that consumption and production sets correspondences are integrably bounded.

Claim 2.5.3. There exists $\bar{x} \in \mathrm{S}^1(X)$ and $\bar{y} \in \mathrm{S}^1(Y)$ such that

$$\text{for a.e. } a \in A \quad \bar{x}(a) = e(a) + \bar{y}(a).$$

Proof. We let $F : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined for all $a \in A$ by $F(a) := X(a) \cap (\{e(a)\} + Y(a))$. The correspondence F is graph measurable with non-empty and closed values. Thus, applying Proposition 2.7.1, there exist $\bar{x} \in \mathrm{S}(X)$ a measurable selection of X and $\bar{y} \in \mathrm{S}(Y)$ a measurable selection of Y such that $\bar{x}(a) = \bar{y}(a)$. We propose to prove that both functions $\bar{x}$ and $\bar{y}$ are integrable. Since Y_Σ is non-empty, there exists $\underline{y} \in \mathrm{S}^1(Y)$. For each $n \in \mathbb{N}$, we let $A^n := \{a \in A \mid \|\bar{y}(a)\| \leqslant n\}$ and we let the function $y^n : A \to \mathbb{L}$ defined by

$$\forall a \in A^n \quad y^n(a) := \bar{y}(a) \quad \text{and} \quad \forall a \in A \setminus A^n \quad y^n(a) = \underline{y}(a).$$

The function y^n is an integrable selection of Y, that is, $y^n \in \mathrm{S}^1(Y)$. For each $n \in \mathbb{N}$, we let $u^n := \int_A y^n(a) d\mu(a)$ and we check that

$$u^n \in Y_\Sigma \cap \left(\left\{ \int_A \inf(\underline{x}(a), \underline{y}(a)) d\mu(a) \right\} + \mathbb{L}_+ \right).$$

[23]Note that for a.e. agent $a \in A$, $x^n(a) \in \text{co}\,G^n(a) \subset \text{co}\,H^n(a)$.

Following Assumption B, $A(Y_\Sigma) \cap \mathbb{L}_+ = \{0\}$ and it follows that the sequence $(u^n)_{n\in\mathbb{N}}$ is bounded. We can suppose (extracting a subsequence if necessary) that $(u^n)_{n\in\mathbb{N}}$ is convergent to $u^* \in Y_\Sigma$. Applying Theorem 2.7.2, there exists an integrable function $\hat{y} : A \to \mathbb{L}$, such that

$$\int_A \hat{y}(a) d\mu(a) \leqslant u^* \quad \text{and} \quad \text{for a.e. } a \in A \quad \hat{y}(a) \in \text{ls}\,\{y^n(a)\}.$$

Since for a.e. $a \in A$, the sequence $(y^n(a))_{n\in\mathbb{N}}$ converges to $\bar{y}(a)$, it follows that $\hat{y} = \bar{y}$. □

We are now ready to construct a sequence of integrably bounded economies. For each $n \in \mathbb{N}$, we let $\mathcal{E}^n$ be the economy

$$\mathcal{E}^n := ((A, \mathcal{A}, \mu), \langle \mathbb{L}^*, \mathbb{L} \rangle, (X^n, Y^n, P^n, e)),$$

where for all $a \in A$,

$$X^n(a) := X(a) \cap K_{\bar{x}}(a, n), \quad Y^n(a) := Y(a) \cap K_{\bar{y}}(a, n)$$

and

$$P^n(a) := P(a) \cap (X^n(a) \times X^n(a))$$

with for each integrable function $z : A \to \mathbb{L}$,

$$K_z(a, n) := \{x \in \mathbb{L} \mid \|x\| \leqslant \max(\|z(a)\|, n)\}.$$

For each $n \in \mathbb{N}$, $\mathcal{E}^n$ satisfies Assumptions C', M', P, S and IB of Lemma 2.5.2. It follows that for each $n \in \mathbb{N}$, there exists $(x^n, p^n) \in \mathrm{S}^1(X) \times \mathbb{L}^*$ with $\|p^n\| = 1$ satisfying $v^n := \int_A x^n \in X_\Sigma \cap (\{\omega\} + Y_\Sigma)$ and such that there exists $A^n \subset A$ with $\mu(A \setminus A^n) = 0$, with for all $a \in A^n$,

$$(x, y) \in P^n_a(x^n(a)) \times Y^n(a) \Longrightarrow p^n(x) \geqslant p^n(e(a)) + p^n(y). \tag{2.6}$$

We can thus suppose (extracting a subsequence if necessary) that $(p^n)_{n\in\mathbb{N}}$ converges to $p^* \in \mathbb{L}^*$ with $\|p^*\| = 1$. Applying Assumption B, we can (extracting a subsequence if necessary) as well assume that the sequence $(v^n)_{n\in\mathbb{N}}$ converges to $v^* \in \{\omega\} + Y_\Sigma$. Applying Theorem 2.7.2, there exists an integrable function $x^* : A \to \mathbb{L}$, such that

$$\int_A x^*(a) d\mu(a) \leqslant v^* \quad \text{and} \quad \text{for a.e. } a \in A \quad x^*(a) \in \text{ls}\,\{x^n(a)\}.$$

Since for a.e. $a \in A$, $X(a)$ is closed, we have that $x^* \in \mathrm{S}^1(X)$, and thus $\int_A x^* - \omega \in Y_\Sigma - \mathbb{L}_+$. Now two cases may occur, production sets are free-disposal (Assumption FD (a)) or preferences are weakly monotone (Assumption FD (b)). We deal with the first situation since the proof of the other one is similar and classic. Assume therefore that the total production set satisfies free-disposal, that is, $-\mathbb{L}_+ \subset A(Y_\Sigma)$. It follows that there exists $y^* \in \mathrm{S}^1(Y)$ such that

$$\int_A x^* = \omega + \int_A y^*.$$

We propose to prove that (x^*, y^*, p^*) is a satiation quasi-equilibrium of $\mathcal{E}$. Condition (ii) of Definition 2.5.1 is already proved. We will now prove condition (i), that is, for almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*(x) \geqslant p^*(e(a)) + p^*(y).$$

Let $a \in A \setminus (\cup_{n\in\mathbb{N}} A^n)$ be such that $P_a(x^*(a)) \neq \emptyset$ and let $(x, y) \in P_a(x^*(a)) \times Y(a)$. For all n large enough, $x^*(a) \in X^n(a)$ and $(x, y) \in P^n_a(x^*(a)) \times Y^n(a)$. We may assume (extracting a subsequence if necessary) that $(x^n(a))_{n\in\mathbb{N}}$ converges to $x^*(a)$. Since P_a is lower semi-continuous, applying (2.6) we get that $p^*(x) \geqslant p^*(e(a)) + p^*(y)$.

2.6 Appendix A : Finitely many agents

2.6.1 The Model and the equilibrium concepts

We consider a production economy with a commodity space $\mathbb{L}$ which is a finite dimensional vector space. The *price-commodity* pairing is modeled by the natural dual pairing $\langle \mathbb{L}^*, \mathbb{L} \rangle$. Let I be the finite set of agents (or consumers). An agent $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{L}$, an initial endowment $e_i \in \mathbb{L}$, a preference relation described by a correspondence P_i from $\prod_{i\in I} X_i$ into X_i and a set $Y_i \subset \mathbb{L}$ representing the production possibilities available to the consumer $i \in I$. A consumption plan x is an element of $\prod_{i\in I} X_i$ and a consumption bundle x_i of agent $i \in I$ is an element of X_i. Consider a consumption plan $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$, for an agent $i \in I$, the set $P_i(x) \subset X_i$ is the set of consumption bundles strictly preferred to x_i by the i-th agent, given the consumption bundles $(x_k)_{k\neq i}$ of the other consumers. The set of production plans is $\prod_{i\in I} Y_i$.

A complete description of a production economy $\mathcal{E}$ is given by the following list:

$$\mathcal{E} := \left(\langle \mathbb{L}^*, \mathbb{L} \rangle , (X_i, Y_i, P_i, e_i)_{i\in I} \right).$$

Definition 2.6.1. An element $(x^*, y^*, p^*) \in \prod_{i\in I} X_i \times \prod_{i\in I} Y_i \times \mathbb{L}^*$ is a *satiation quasi-equilibrium* of an economy $\mathcal{E}$, if $p^* \neq 0$ and if the following properties are satisfied.

(i) For every $i \in I$,

$$(x_i, y_i) \in P_i(x^*) \times Y_i \Longrightarrow p^*(x_i) \geqslant p^*(y_i) + p^*(e_i).$$

(ii)

$$\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{i\in I} y_i^*.$$

Remark 2.6.1. Note that the concept of satiation quasi-equilibrium is closely related to the concept of Edgeworth equilibrium. Indeed, following Florenzano [16] and [17], if $x^* \in \prod_{i\in I} X_i$ is an Edgeworth equilibrium decentralized by a price $p^* \in \mathbb{L}^*$ then there exists a production plan $y^* \in \prod_{i\in I} Y_i$ such that (x^*, y^*, p^*) is a satiation quasi-equilibrium.

In order to prove the existence of satiation quasi-equilibria, we now present the list of assumptions that economies will be required to satisfy.

2.6.2 The Assumptions

Assumption (C$_f$). *For each agent $i \in I$, X_i is closed convex, P_i is lower semi continuous and for each consumption plan $x = (x_i) \in \prod_{i\in I} X_i$, $x_i \notin \operatorname{co} P_i(x)$.*

Assumption (P$_f$). *For each agent $i \in I$, the production set Y_i is closed convex.*

Assumption (B$_f$). *For each agent $i \in I$, the consumption set X_i and the production set Y_i are bounded.*

Assumption (S$_f$). *For each agent $i \in I$, $e_i \in X_i - Y_i$.*

2.6.3 Existence Result

Theorem 2.6.1. *If $\mathcal{E}$ is an economy with finitely many consumers satisfying Assumptions C_f, P_f, B_f and S_f, then there exists a satiation quasi-equilibrium.*

Proof. The proof follows almost verbatim the proof of Theorem 2.1.1 in Oiko Nomia [27]. Without any loss of generality we can suppose that for each agent $i \in I$, $e_i = 0$ and for each $x \in \prod_{i\in I} X_i$, $P_i(x)$ is convex. Following Assumption S there exists $(\hat{x}, \hat{y}) \in \prod_{i\in I} X_i \times \prod_{i\in I} Y_i$ such that $\sum_{i\in I} \hat{x}_i = \sum_{i\in I} \hat{y}_i$. We consider a norm $\|.\|$ on $\mathbb{L}$ and we let $\Delta = \{p \in \mathbb{L}^* \mid \|p\|^* \leqslant 1\}$. For each $p \in \Delta$ and each agent $i \in I$, we let $\pi_i(p) := \sup\{p(z_i) \mid z_i \in Y_i\}$, $B_i(p) := \{z_i \in X_i \mid p(z_i) \leqslant \pi_i(p) + (1 - \|p\|)\}$, $A_i(p) := \{z_i \in X_i \mid p(z_i) < \pi_i(p) + (1 - \|p\|)\}$ and

$$\Gamma_i(p) = \begin{cases} \{\hat{x}_i\} & \text{if} \quad A_i(p) = \emptyset \\ B_i(p) & \text{otherwise.} \end{cases}$$

For each $(x, y, p) \in \prod_{i \in I} X_i \times \prod_{i \in I} Y_i \times \Delta$ and for each agent $i \in I$, we let

$$\phi_i(x, y, p) \quad := \quad \begin{cases} \Gamma_i(p) & \text{if} \quad x_i \notin B_i(p) \\ P_i(x) \cap A_i(p) & \text{if} \quad x_i \in B_i(p)\ , \end{cases}$$

$$\psi_i(x, y, p) \quad := \quad \{z_i \in Y_i \mid p(z_i) > p(y_i)\}\ ,$$

$$\theta(x, y, p) \quad := \quad \left\{q \in \Delta \mid q\left(\textstyle\sum_{i \in I}(x_i - y_i)\right) > p\left(\textstyle\sum_{i \in I}(x_i - y_i)\right)\right\}.$$

Following Oiko Nomia [27], we apply a fixed point theorem (Gale and Mas-Colell [18]) to provide the existence of (x^*, y^*, p^*) such that for each $i \in I$, $\phi_i(x^*, y^*, p^*) = \emptyset$, $\psi_i(x^*, y^*, p^*) = \emptyset$ and $\theta(x^*, y^*, p^*) = \emptyset$.

We let $u^* := \sum_{i \in I}(x_i^* - y_i^*)$. If $p^* = 0$ then for each $i \in I$, $A_i(p^*) = X_i$ and thus $P_i(x^*) = \emptyset$. Moreover for each $q \in \Delta$, $q(u^*) \leqslant 0$. It follows that $u^* = 0$ and (x^*, y^*, q) is a satiation quasi-equilibrium for all [24] $q \in \mathbb{L}^*$ with $q \neq 0$.

If $p^* \neq 0$ then for each $i \in I$, $p^*(y_i^*) = \sup\{p^*(y_i) \mid y_i \in Y_i\}$ and for each $x_i \in P_i(x^*)$, $p^*(x_i) \geqslant p^*(y_i^*)$. It remains to prove that $u^* = 0$. If $\|p^*\|^* < 1$ then there exists a neighborhood V of 0 in $\mathbb{L}^*$ such that for all $q \in V$, $q(u^*) \leqslant 0$. It follows that $u^* = 0$. Now if $\|p^*\|^* = 1$ then for all $q \in \Delta$, $q(u^*) \leqslant p^*(u^*)$. But for each $i \in I$, $x_i^* \in B_i(p^*)$ and then $p^*(u^*) \leqslant 0$. It follows that $u^* = 0$ and (x^*, y^*, p^*) is a satiation quasi-equilibrium. □

2.7 Appendix B : Measurability and integration of correspondences

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a complete separable metric space.

2.7.1 Measurability of correspondences

A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a, x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$.

Following Castaing and Valadier [9] and Himmelberg [23], we recall the two following classical characterizations of measurable correspondences.

Proposition 2.7.1. *Consider $F : A \twoheadrightarrow D$ a correspondence with non-empty closed values. The following properties are equivalent.*

(i) *The correspondence F is measurable.*

(ii) *There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable selections of F such that for all $a \in A$, $F(a) = \mathrm{cl}\,\{f_n(a) \mid n \in \mathbb{N}\}$.*

(iii) *For each $x \in D$, the function $\delta_F(., x) : a \mapsto d(x, F(a))$ is measurable.*

[24] Since each consumer is satiated, the concept of value is obsolete.

Proposition 2.7.2. *Consider $F : A \twoheadrightarrow D$ a correspondence.*

(i) *If F has non-empty closed values then the measurability of F implies the graph measurability of F.*

(ii) *If $(A, \mathcal{A}, \mu)$ is complete then the graph measurability of F implies the measurability of F.*

(iii) *If F has non-empty closed values and $(A, \mathcal{A}, \mu)$ is complete then measurability and graph measurability of F are equivalent.*

Following Aumann [5], graph measurable correspondences (possibly without closed values) have measurable selections.

Proposition 2.7.3. *Consider F a graph measurable correspondence from A into D with non-empty values. If $(A, \mathcal{A}, \mu)$ is complete then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of measurable selections of F, such that for all $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ is dense in $F(a)$.*

2.7.2 Measurability of preference relations

Let P be a correspondence from A into $D \times D$. For each function $x : A \to D$ the *upper section relative to x* is noted $P_x : A \twoheadrightarrow D$ and is defined by $a \mapsto \{y \in D \mid (x(a), y) \in P(a)\}$. For each function $y : A \to D$ the *lower section relative to y* is noted $P^y : A \twoheadrightarrow D$ and is defined by $a \mapsto \{x \in D \mid (x, y(a)) \in P(a)\}$.

Let $X : A \twoheadrightarrow D$ be a correspondence. A *correspondence of preference relations in X* is a correspondence P from A into $D \times D$ satisfying for all $a \in A$, $P(a) \subset X(a) \times X(a)$. For each $a \in A$, we note P_a the correspondence [25] from $X(a)$ into $X(a)$ defined by $x \mapsto \{y \in X(a) \mid (x, y) \in P(a)\}$. For each $y \in X(a)$ the lower inverse image of y by P_a is noted $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$. The correspondence of preference relations P is graph measurable if

$$\{(a, x, y) \in A \times D \times D \mid (x, y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D) \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *Aumann measurable* if

$$\forall (x, y) \in S(X) \times S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

The correspondence of preference relations P in X is *lower graph measurable* if for each measurable selection y of X, the correspondence P^y is graph measurable, that is

$$\forall y \in S(X) \quad G_{P^y} = \{(a, x) \in A \times D \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *upper graph measurable* if for each measurable selection x of X, the correspondence P_x is graph measurable, that is

$$\forall x \in S(X) \quad G_{P_x} = \{(a, y) \in A \times D \mid (x(a), y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

We propose to compare these three concepts of measurability of preference relations.

Proposition 2.7.4. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Then the graph measurability of P implies the lower and upper graph measurability of P, and lower or upper graph measurability of P implies the Aumann measurability of P.*

Proof. This is a direct consequence of the Projection Theorem in Castaing and Valadier [9]. □

Under additional assumptions, the converse is true.

[25] Remark that the graph of P_a and $P(a)$ coincide.

Proposition 2.7.5. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Moreover, we suppose that for a.e. $a \in A$, $X(a)$ is a closed connected subset of D, $P(a)$ is an irreflexive and transitive binary relation on $X(a)$ and for each $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$. If at least one of the two following properties holds,*

1. *for a.e. $a \in A$, $X(a) = (\mathbb{R}_+)^\ell$ where [26] $D = \mathbb{R}^\ell$ and $P(a)$ is strictly monotone [27],*

2. *for a.e. $a \in A$, $P(a)$ is negatively transitive,*

then the Aumann measurability of P implies the lower and upper graph measurability of P, and the lower and upper graph measurability of P implies the graph measurability of P.

Remark 2.7.1. In Aumann [4] and Schmeidler [30], Property 1 is satisfied. In Hildenbrand [21] , for all $a \in A$, $P(a)$ is ordered and then property 2 is satisfied.

Proof. Suppose that P is Aumann measurable. We distinguish two cases. Under Property 1, $(\mathbb{Q}_+)^\ell$ is dense in $X(a)$ for all $a \in A$, hence if $(x, y) \in P(a)$ then there exists $r \in (\mathbb{Q}_+)^\ell$ such that $(x, r) \in P(a)$ and $r < y$. It follows that, if $x \in S(X)$ is a measurable selection of X, then

$$G_{P_x} = \bigcup_{r \in \mathbb{Q}_+^\ell} \left(\{(a \in A \mid (x(a), r) \in P(a)\} \times (\mathbb{R}_+)^\ell \right) \cap (A \times \{y \in D \mid r < y\})$$

and $G_{P_x} \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$. Similarly we can prove that $G_{P^x} \in \mathcal{A} \times \mathcal{B}(\mathbb{R}^\ell)$.

Under Property 2, to prove that P is both upper and lower graph measurable, we can follow almost verbatim the proof of Lemma in Appendix in Podczeck [28]. The graph of X is measurable, then Proposition 2.7.2 implies that X has a Castaing representation, that is there exists a sequence $(h_i)_{i \in \mathbb{N}}$ of measurable selections of X, such that for all $a \in A$, $X(a) = \operatorname{cl}\{h_i(a) \mid i \in \mathbb{N}\}$. Now suppose that a measurable selection $x \in S(X)$ has been given. Consider any $a \in A$ and let $y \in X(a)$. If $(x(a), y) \in P(a)$, then following Debreu [12], there exists $i \in \mathbb{N}$ such that $(x(a), h_i(a)) \in P(a)$ and $(h_i(a), y) \in P(a)$. By the continuity of $P(a)$, for each $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $d(y, h_j(a)) \leqslant 1/n$ and $(h_i(a), h_j(a)) \in P(a)$. Conversely, if for some $i \in \mathbb{N}$, $(x(a), h_i(a)) \in P(a)$ and for each $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $d(y, h_j(a)) \leqslant 1/n$ and $(h_i(a), h_j(a)) \in P(a)$, then $y \in \operatorname{cl} P_a(h_i(a)) \subset P_a(x(a))$. It follows that

$$G_{P_x} = G_X \cap \bigcup_{i \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} [(A(i,j) \times D) \cap \{(a, y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\}] \ ,$$

where

$$A(i,j) = \{a \in A \mid (x(a), h_i(a)) \in P(a)\} \cap \{a \in A \mid (h_i(a), h_j(a)) \in P(a)\}.$$

Since P is Aumann measurable, for each $(i, j) \in \mathbb{N}^2$, $A(i,j) \in \mathcal{A}$. Finally following [9] or [23], for each $(j, n) \in \mathbb{N}^2$, $\{(a, y) \in A \times D \mid d(a, h_j(a)) \leqslant 1/n\} \in \mathcal{A} \times \mathcal{B}(D)$, and P is upper graph measurable. Similarly we prove that P is lower graph measurable.

Suppose now that P is upper and lower graph measurable. Let $(a, x, y) \in G_P$, that is $(x, y) \in P(a)$. We distinguish two cases. Under property 2 there exists $i \in \mathbb{N}$ such that

$$(x, h_i(a)) \in P(a) \quad \text{and} \quad (h_i(a), y) \in P(a).$$

Since $P(a)$ is transitive, the converse is true, and

$$G_P = \bigcup_{i \in \mathbb{N}} \left\{(a, x, y) \in A \times D \times D \mid (a, x) \in G_{P(.,h_i(.))} \quad \text{and} \quad (a, y) \in G_{P^{-1}(.,h_i(.))}\right\}.$$

It follows that P is graph measurable.

[26] For some integer $\ell \in \mathbb{N}$.

[27] That is for all $x \in X(a)$, for all $m \in (\mathbb{R}_+)^\ell$, $x + m \in P_a(x) \cup \{x\}$.

Under property 1 there exists $r \in (\mathbb{Q}_+)^\ell$ such that $(x, r) \in P(a)$ and $r < y$. Since preference relations are monotone the converse is true and

$$G_P = \bigcup_{r \in (\mathbb{Q}_+)^\ell} \{(a, x, y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell \mid (x, r) \in P(a)\} \times \{(a, x, y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell \mid r < y\}.$$

It follows that P is graph measurable. □

We recall that the correspondence P_a is lower semi-continous if for all open set $V \subset D$, $\{x \in X(a) \mid P_a(x) \cap V \neq \emptyset\}$ is open in $X(a)$.

We introduce a notion of measurability of preference relations, close to the notion of lower semi-continuity.

Definition 2.7.1. The correspondence of preference relations P in X is lower semi-graph measurable if for each graph measurable correspondence $V : A \twoheadrightarrow D$ with open values, the following set is measurable

$$\{(a, x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} \in \mathcal{A} \times \mathcal{B}(D).$$

We propose to compare this measurability notion with the other notions introduced before.

Proposition 2.7.6. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph.*

(i) *Graph measurability of P implies the lower semi-graph measurability of P.*

(ii) *If for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is open in $X(a)$, then the lower graph measurability of P implies the lower semi-graph measurability of P.*

(iii) *If for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is closed in $X(a)$, then the lower semi-graph measurability of P implies the lower graph measurability of P.*

Proof. The part (i) is a direct consequence of Projection Theorem in Castaing and Valadier [9]. Indeed,

$$\{(a, x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} = \pi\left[G_P \cap \{(a, x, y) \in A \times D \times D \mid y \in V(a)\}\right],$$

where $\pi : A \times D \times D \to A \times D$ is the projection $(a, x, y) \mapsto (a, x)$.

Suppose now that the correspondence P is lower graph measurable and that for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is open in $X(a)$. Let $(a, x) \in G_X$ such that $P_a(x) \cap V(a) \neq \emptyset$. Following Proposition 2.7.3, there exists a sequence $(z_n)_{n \in \mathbb{N}}$ of measurable selections of X, such that for all $a \in A$, $(z_n(a))_{n \in \mathbb{N}}$ is dense in $X(a)$. The set $P_a(x) \cap V(a)$ is open in $X(a)$, it follows that there exists $n \in \mathbb{N}$ such that $z_n(a) \in P_a(x) \cap V(a)$. The converse is true and then

$$\{(a, x) \in G_X \mid P_a(x) \cap V(a) \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} \left[G_{P^{z_n}} \cap (\{a \in A \mid z_n(a) \in V(a)\} \times D)\right].$$

That is P is lower semi-graph measurable.

Suppose now that the correspondence P is lower semi-graph measurable and that for a.e. $a \in A$, for all $x \in X(a)$, $P_a(x)$ is closed in $X(a)$. Let $y \in S(X)$ be a measurable selection of X. Let $(a, x) \in G_{P^y}$, that is, $x \in X(a)$ and $y \in P_a(x)$. Let $n \in \mathbb{N}$, and consider $V_n(a) := \{z \in D \mid d(z, y(a)) < 1/(n+1)\}$. Then for all $n \in \mathbb{N}$, $P_a(x) \cap V_n(a) \neq \emptyset$. Conversely, is for all $n \in \mathbb{N}$, $P_a(x) \cap V_n(a) \neq \emptyset$, then $y(a) \in \operatorname{cl} P_a(x)$. Since $P_a(x)$ is closed in $X(a)$, then $(a, x) \in G_{P^y}$. Thus

$$G_{P^y} = \bigcap_{n \in \mathbb{N}} \{(a, x) \in G_X \mid P_a(x) \cap V_n(a) \neq \emptyset\}.$$

And the correspondence P is lower graph measurable. □

2.7.3 Integration of correspondences

We suppose in this section that $(A, \mathcal{A}, \mu)$ is a finite complete measure space. If $F : A \twoheadrightarrow \mathbb{L}$ is a correspondence from A to $\mathbb{L}$, the set of integrable selections of F is noted $S^1(F)$. We note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a)d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in S^1(F) \quad v = \int_A x(a)d\mu(a) \right\}$.

Proposition 2.7.7. *Consider $F : A \twoheadrightarrow \mathbb{L}$ a graph measurable correspondence. If F_Σ is non-empty, we let $G : A \twoheadrightarrow \mathbb{L}$ the correspondence defined by*

$$\forall a \in A \quad G(a) := \mathrm{cl}\left[\overline{\mathrm{co}}\, F(a) + A(F_\Sigma)\right].$$

If F_Σ is non-empty, closed and convex then $G_\Sigma = F_\Sigma$ and for all $p \in \mathbb{L}^*$, if there exists an integrable selection g^* of G such that for a.e. $a \in A$, $p\,(g^*(a)) = \sup p\,(G(a))$, then there exists an integrable selection f^* of F satisfying for a.e. $a \in A$, $p\,(f^*(a)) = \sup p\,(F(a))$ and $\int_A f^* = \int_A g^*$.

Proof. Since $(A, \mathcal{A}, \mu)$ is complete, following Proposition 2.7.2, the correspondence F is measurable. Following Rockafellar and Wets [29], the correspondence G is measurable with closed-values. Once again applying Proposition 2.7.2, G is graph measurable and $F_\Sigma \subset G_\Sigma$. Moreover if $p \in \mathbb{L}$ then for all $a \in A$, $\sup p\,(G(a)) = \sup p\,(F(a)) + \sup p\,(A(F_\Sigma))$. Note that, since $A(F_\Sigma)$ is a cone containing zero, $\sup p\,(A(F_\Sigma)) \in \{0, +\infty\}$.

Suppose now that that F_Σ is non-empty, closed and convex, and suppose that there exists $v \in G_\Sigma$ such that $v \notin F_\Sigma$. Since F_Σ is closed convex, by a separation argument there exists $p \in \mathbb{L} \setminus \{0\}$ such that $p\,(v) > \sup p\,(F_\Sigma)$. It follows that $\sup p\,(A(F_\Sigma)) = 0$ and following Theorem C in Hildenbrand [21],

$$\sup p\,(F_\Sigma) = \int_A \sup p\,(F(a))\, d\mu(a) = \int_A \sup p\,(G(a))\, d\mu(a) = \sup p\,(G_\Sigma)\,.$$

Thus $p\,(v) > \sup p\,(G_\Sigma)$ and this contradicts the fact that $v \in G_\Sigma$. The rest of the proof of Proposition 2.7.7 is a direct consequence of this result. □

We are now ready to present two versions of Fatou's Lemma in several dimensions. The first one is due to Artstein [2].

Theorem 2.7.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from A to $\mathbb{L}$, integrably bounded and such that $\lim_{n\to\infty} \int_A f_n$ exists. Then there exists an integrable function f from A to $\mathbb{L}$ such that*

$$\int_A f = \lim_{n\to\infty} \int_A f_n \quad \textit{and} \quad \textit{for a.e. } a \in A \quad f(a) \in \mathit{ls}\{f_n(a)\}.$$

The second one is due to Cornet and Topuzu [11]. This version of Fatou's Lemma generalizes a version of Schmeidler [31] to more general positive cones.

Theorem 2.7.2. *Let $C \subset \mathbb{L}$ be a pointed closed convex cone. We note $\geqslant$ the partial order induced [28] by C. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from A to $\mathbb{L}$ integrably bounded from below [29] and such that $\lim_{n\to\infty} \int_A f_n$ exists. Then there exists an integrable function f from A to $\mathbb{L}$ such that*

$$\int_A f \leqslant \lim_{n\to\infty} \int_A f_n \quad \textit{and} \quad \textit{for a.e. } a \in A \quad f(a) \in \mathit{ls}\{f_n(a)\}.$$

For related results we refer to Balder [6] and Balder and Hess [8].

[28] For all $(x, y) \in \mathbb{L}^2$, $x \geqslant y$ whenever $x - y \in C$.
[29] That is, there exists an integrable function g such that for each $n \in \mathbb{N}$, for almost every $a \in A$, $f_n(a) \geqslant g(a)$.

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Chapitre 3

Existence d'équilibres avec double infinité et des préférences non ordonnées

Résumé

L'approche introduite dans le chapitre précédent est maintenant appliquée pour démontrer l'existence d'un équilibre de Walras pour des économies avec une double infinité d'agents et de biens. L'espace des biens est modélisé par un espace de Banach séparable ordonné par un cône d'intérieur non-vide. Notre approche, basée sur la discrétisation des correspondances (ou multifonctions) mesurables, nous permet de démontrer l'existence d'un équilibre aussi bien pour des économies avec des préférences non ordonnées mais convexes que pour des économies avec des préférences ordonnées mais non convexes. Notre résultat d'existence généralise le théorème 5.1 de Podczeck [20] et complète le théorème d'existence de Khan and Yannelis [18].

Mots-clés : *Espace mesuré d'agents, espace de Banach séparable, préférences non ordonnées mais convexes, préférences ordonnées mais non convexes et discrétisation des correspondances mesurables.*

Existence of equilibria for large square economies with non-ordered preferences

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16th June 2002

Abstract

The Approach of Martins Da Rocha [19] is applied to provide a Walrasian equilibria existence result for economies with a measure space of agents and a large commodity space. The commodity space is modeled by an ordered separable Banach space whose positive cone has a non-empty interior. The approach proposed in this paper, based on the discretization of measurable correspondences, allows us to provide an existence result (Theorem 3.3.1) for economies with a non-trivial production sector and with possibly non-ordered but convex preferences, as well as partially ordered (possibly incomplete) but non-convex preferences. This result completes the Main Theorem in Khan and Yannelis [18] and generalizes Theorem 5.1 in Podczeck [20].

Keywords : *Measure space of agents, separable Banach commodity spaces, non-ordered but convex preferences, partialy ordered but non-convex preferences and discretization of measurable correspondences.*

3.1 Introduction

For economies with a measure space of agents and an ordered separable [1] Banach commodity space, there exist many Walrasian equilibria existence results for exchange economies with ordered preferences. In Khan and Yannelis [18], the preferences are ordered and convex. In Rustichini and Yannelis [22] or in Podczeck [20], the preferences are ordered but non-convex.

The *discretization* approach proposed in this paper, enables us to provide an existence result (Theorem 3.3.1) for economies with a non-trivial production sector and with possibly non-ordered but convex preferences as well as partially ordered (possibly incomplete) but non-convex preferences. Theorem 3.3.1 completes Main Theorem in Khan and Yannelis [18] and generalizes Theorem 5.1 in Podczeck [20].

The *discretization* approach consists on considering an economy with a measure space of agents as the *limit* of a sequence of economies with a finite, but larger and larger, set of agents. We construct a sequence of partitions of the measure space depending on the characteristics of the economy. To each partition we define a *subordinated simple* economy. Each *simple* economy will be identified as an economy with a finite set of agents, and applying a classical Edgeworth equilibria existence result for economies with a finite set of agents and a large commodity space (e.g. Florenzano [14]), we get a sequence of allocations and prices, which will converge to a Walrasian quasi-equilibrium for the original economy.

The paper is organized as follows. In Section 3.2, we set the main definitions and notations. In Section 3.3 we define the model of large square economies, we introduce the concepts of equilibria, we give the list of assumptions that economies will be required to satisfy and finally, we present the existence result (Theorem 3.3.1). The Section 3.4 is devoted to the mathematical *discretization* of measurable correspondences. The proof of the main theorem (Theorem 3.3.1) is then given in Section 3.5. The last section is devoted to mathematical auxiliary results.

3.2 Notations and definitions

Consider (E, τ) a topological vector space. If $X \subset E$ is a subset, then the τ-interior of X is noted $\tau\text{-int}\, X$, the τ closure of X is noted $\tau\text{-cl}\, X$. The convex hull of X is noted $\operatorname{co} X$ and the τ closed convex hull of X is noted $\tau\text{-}\overline{\operatorname{co}}\, X$. If X is convex then we let $A(X) = \{v \in \mathbb{L} \mid X + \{v\} \subset X\}$ be

[1] In [24], Tourky and Yannelis proved that equilibria existence results in [18] and [22] do not extend to non-separable commodity spaces.

the asymptotic cone of X. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of subsets of E, the τ *sequential upper limit* of $(C_n)_{n\in\mathbb{N}}$, is denoted $\tau\text{-ls}\, C_n$ and is defined by

$$\tau\text{-ls}\, C_n := \{x \in E \mid x = \tau\text{-}\lim x_k\,, \quad x_k \in C_{n(k)}\}.$$

Let $(\mathbb{L}, \|.\|, \geqslant)$ be an ordered separable Banach space [2]. The topology induced by the norm is noted s (strong). The s-dual of $\mathbb{L}$, that is, the space of s-continuous linear functionals on $\mathbb{L}$, is noted $\mathbb{L}'$. The natural dual pairing $\langle \mathbb{L}', \mathbb{L}\rangle$ is defined by $\langle p, x\rangle := p(x)$, for all $(p, x) \in \mathbb{L}' \times \mathbb{L}$. The weak topology $\sigma(\mathbb{L}, \mathbb{L}')$ is noted w and the weak star topology $\sigma(\mathbb{L}', \mathbb{L})$ is noted w^*. The space $\mathbb{L}$ is thus endowed with two topologies s and w. Following Podczeck [20], the Borel σ-algebra of $(\mathbb{L}, w)$ and of $(\mathbb{L}, s)$ coincide and is noted $\mathcal{B}(\mathbb{L})$. The positive cone of $\mathbb{L}$ is noted $\mathbb{L}_+ := \{x \in \mathbb{L} \mid x \geqslant 0\}$. We write $\mathbb{L}'_+$ for the set $\{p \in \mathbb{L}' \mid \forall x \in \mathbb{L}_+ \quad p(x) \geqslant 0\}$. If $x \in \mathbb{L}$ then $x > 0$ means $x \geqslant 0$ and $x \neq 0$. If $p \in \mathbb{L}'$ then $p > 0$ means $p \geqslant 0$ and $p \neq 0$.

We consider $(A, \mathcal{A}, \mu)$ a finite measure space, that is, A is a set, $\mathcal{A}$ is a σ-algebra of subsets of A and μ is a finite measure on $\mathcal{A}$. The measure space $(A, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all μ-negligible [3] subsets of A. A function f from A to $\mathbb{L}$ is *measurable* if for all $B \in \mathcal{B}(\mathbb{L})$, $f^{-1}(B) := \{a \in A \mid f(a) \in B\} \in \mathcal{A}$. A function f from A to $\mathbb{L}$ is *Bochner measurable* if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ pointwise s-converging to f, that is,

$$\forall a \in A \qquad \lim_{n\to\infty} \|f_n(a) - f(a)\| = 0.$$

Since $(\mathbb{L}, \|.\|)$ is separable then following Theorem 4.38 in Aliprantis and Border [1], f is measurable if and only if f is Bochner measurable. A measurable function f from A to $\mathbb{L}$ is *Bochner integrable* if the real-valued function $\|f(.)\|$ is integrable. Following Diestel and Uhl [11], a measurable function f is Bochner integrable if and only if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \int_A \|f_n(a) - f(a)\|\, d\mu(a) = 0.$$

For each $E \in \mathcal{A}$, the integral of f over E is defined by

$$\int_E f(a) d\mu(a) := \lim_{n\to\infty} \int_E f_n(a) d\mu(a).$$

Let (D, d) be a separable metric space. A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a, x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$. When $D \subset \mathbb{L}$ the set of Bochner integrable selections of F is noted $S^1(F)$ and we note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a) d\mu(a) := \big\{v \in D \bigm| \exists x \in S^1(F) \quad v = \int_A x(a) d\mu(a)\big\}$. The correspondence F is said to be *integrably bounded* if there exists an integrable function h from A to $\mathbb{R}_+$ such that for a.e. $a \in A$, for all $x \in F(a)$, $\|x\| \leqslant h(a)$.

Let X be a space and $P \subset X \times X$ be a binary relation on X. The relation P is irreflexive if $(x, x) \notin P$, for all $x \in X$. The relation P is transitive if $[(x, y) \in P$ and $(y, z) \in P]$ implies $(x, z) \in P$, for all $(x, y, z) \in X^3$. The relation P is negatively transitive if $[(x, y) \notin P$ and $(y, z) \notin P]$ implies $(x, z) \notin P$, for all $(x, y, z) \in X^3$. The relation P is a partial order it is irreflexive and transitive. The relation P is an order if it is irreflexive, transitive and negatively transitive. When P is an order, it is usually noted $\succ$ and $X^2 \setminus P$ is noted $\preceq$. Note that when P is an order, then $\preceq$ is transitive, reflexive ($x \preceq x$ for all $x \in X$) and complete (for all $(x, y) \in X^2$ either $x \preceq y$ or $y \preceq x$).

[2] That is $(\mathbb{L}, \|.\|)$ is a separable Banach space and there exists a pointed $(C \cap -C = \{0\})$ closed convex cone $C \subset \mathbb{L}$ such that $\geqslant$ is the order induced by C, that is $x \geqslant y$ whenever $x - y \in C$.

[3] A set N is μ-negligible if there exists $E \in \mathcal{A}$ such that $N \subset E$ and $\mu(E) = 0$.

3.3 The model, the equilibrium concepts and the assumptions

3.3.1 The Model

We consider an ordered separable Banach space $(\mathbb{L}, \|.\|, \geqslant)$ such that the positive cone $\mathbb{L}_+ := \{x \in \mathbb{L} \mid x \geqslant 0\}$ is closed and has a non-empty s-interior. Moreover, we consider a complete finite measure space $(A, \mathcal{A}, \mu)$, a Bochner integrable function e from A to $\mathbb{L}$, two correspondences X and Y from A into $\mathbb{L}$ and a correspondence of preferences P in X, that is, P is a correspondence from A into $\mathbb{L} \times \mathbb{L}$ such that for all $a \in A$, $P(a) \subset X(a) \times X(a)$ and $P(a)$ is an irreflexive relation on $X(a)$.

A large square economy $\mathcal{E}$ is a list

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, Y, P, e)).$$

The commodity space is represented by $\mathbb{L}$. The natural dual pairing $\langle \mathbb{L}', \mathbb{L} \rangle$ is interpreted as the *price-commodity* pairing.

The set of agents (or consumers) is represented by A, the set $\mathcal{A}$ represents the set of admissible coalitions, and the number $\mu(E)$ represents the fraction of consumers which are in the coalition $E \in \mathcal{A}$.

For each agent $a \in A$, the consumption set is represented by $X(a) \subset \mathbb{L}$ and the preference relation is represented by $P(a) \subset X(a) \times X(a)$. We define the correspondence [4] $P_a : X(a) \twoheadrightarrow X(a)$ by $P_a(x) = \{x' \in X(a) \mid (x, x') \in P(a)\}$. In particular, if $x \in X(a)$ is a consumption bundle, $P_a(x)$ is the set of consumption bundles strictly preferred to x by the agent a. The set of consumption allocations (or plans) of the economy is the set $S^1(X)$ of Bochner integrable selections of X. The aggregate consumption set X_Σ is defined by

$$X_\Sigma := \int_A X(a) d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in S^1(X) \quad v = \int_A x(a) d\mu(a) \right\}.$$

The initial endowment of the consumer $a \in A$ is represented by the commodity bundle $e(a) \in \mathbb{L}$. We note $\omega := \int_A e(a) d\mu(a)$ the aggregate initial endowment. The production possibilities available to the consumer $a \in A$ are represented by the set $Y(a) \subset \mathbb{L}$. The set of production allocations (or plans) of the economy is the set $S^1(Y)$ of Bochner integrable selections of Y. The aggregate production set Y_Σ is defined by

$$Y_\Sigma := \int_A Y(a) d\mu(a) := \left\{ u \in \mathbb{L} \;\middle|\; \exists y \in S^1(Y) \quad u = \int_A y(a) d\mu(a) \right\}.$$

3.3.2 The Equilibrium Concepts

Definition 3.3.1. A *Walrasian equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $S^1(X) \times S^1(Y) \times \mathbb{L}'$ such that $p^* \neq 0$ and satisfying the following properties.

(a) For almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a)) \quad \text{and} \quad x \in P_a(x^*(a)) \Longrightarrow p^*(x) > p^*(x^*(a)).$$

(b) For almost every $a \in A$,

$$y \in Y(a) \Longrightarrow p^*(y) \leqslant p^*(y^*(a)).$$

(c)

$$\int_A x^*(a) d\mu(a) = \int_A e(a) d\mu(a) + \int_A y^*(a) d\mu(a).$$

A *Walrasian quasi-equilibrium* of an economy $\mathcal{E}$ is an element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ such that $p^* \neq 0$ and which satisfies conditions (b) and (c) together with

[4] Note that the binary relation $P(a)$ coincide with the graph of the correspondence P_a.

(a') for almost every $a \in A$,

$$p^*\left(x^*(a)\right) = p^*\left(e(a)\right) + p^*\left(y^*(a)\right) \quad \text{and} \quad x \in P_a(x^*(a)) \Longrightarrow p^*\left(x\right) \geqslant p^*\left(x^*(a)\right).$$

Following Debreu [9], we introduce the concept of free-disposal equilibria.

Definition 3.3.2. An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ is a *free-disposal equilibrium* of an economy $\mathcal{E}$ if $p^* > 0$ and if conditions (a) and (b) together with the following condition satisfied.

(c')

$$\int_A x^*(a)d\mu(a) \leqslant \int_A e(a)d\mu(a) + \int_A y^*(a)d\mu(a).$$

An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ is a *free-disposal quasi-equilibrium* of an economy $\mathcal{E}$ if $p^* > 0$ and conditions (a'), (b) and (c') are satisfied.

A (free-disposal) Walrasian equilibrium of a production economy $\mathcal{E}$ is clearly a (resp. free-disposal) Walrasian quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, a classical condition on $\mathcal{E}$ under which a (free-disposal) Walrasian quasi-equilibrium is in fact a (resp. free-disposal) Walrasian equilibrium.

Remark 3.3.1. Every (free-disposal) Walrasian quasi-equilibrium (x^*, y^*, p^*) of $\mathcal{E}$, is a (resp. free-disposal) Walrasian equilibrium, if we assume that, for almost every agent $a \in A$, $X(a)$ is convex, the strict-preferred set $P_a(x^*(a))$ is s-open in $X(a)$ and

$$\inf p^*\left(X(a)\right) < p^*\left(e(a)\right) + \sup p^*\left(Y(a)\right).$$

In particular, if $p^* > 0$ then the last condition is automatically valid if for almost every agent $a \in A$,

$$\Big(\{e(a)\} + Y(a) - X(a)\Big) \cap s - \operatorname{int} \mathbb{L}_+ \neq \emptyset.$$

A Walrasian equilibrium (quasi-equilibrium) of a production economy $\mathcal{E}$ is clearly a free-disposal equilibrium (resp. quasi-equilibrium) of $\mathcal{E}$. We provide in the following remark, a classical condition on $\mathcal{E}$ under which a free-disposal equilibrium (quasi-equilibrium) is in fact a equilibrium (resp. quasi-equilibrium).

Remark 3.3.2. If the aggregate production set Y_Σ is free-disposal, that is, $-\mathbb{L}_+ \subset A(Y_\Sigma)$, then each free-disposal equilibrium (quasi-equilibrium) is in fact a Walrasian (resp. quasi-equilibrium) equilibrium.

Remark 3.3.3. We can find in the literature a third concept of equilibrium. In Khan and Yannelis [18] and Rustichini and Yannelis [22], (x^*, y^*, p^*) with $p^* > 0$, is a *competitive equilibrium* of $\mathcal{E}$ if it satisfies the conditions (b), (c') together with the following (a")

(a") For almost every $a \in A$,

$$p^*\left(x^*(a)\right) \leqslant p^*\left(e(a)\right) + p^*\left(y^*(a)\right)$$

and

$$x \in P_a(x^*(a)) \Rightarrow p^*\left(x\right) > p^*\left(e(a)\right) + p^*\left(y^*(a)\right).$$

The free-disposal property on the aggregate production set is not strong enough to prove that a competitive equilibrium is in fact a Walrasian equilibrium. However, under a suitable *Local Non-Satiation* property and together with the free-disposal property on the aggregate production set, we can prove that a competitive equilibrium is in fact a Walrasian equilibrium. Note moreover that if (x^*, y^*, p^*) is a free-disposal equilibrium then the value of the excess of demand is zero, that is $p^*\left(\int_A y^*(a)d\mu(a) + \omega - \int_A x^*(a)d\mu(a)\right) = 0$. It is not automatically the case if (x^*, y^*, p^*) is a competitive equilibrium.

The model of production economies defined above encompasses the two models presented in Hildenbrand [15].

In a *private ownership economy* $\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, P, e), (Y_j, \theta_j)_{j\in J})$, the production sector is represented by a finite set J of firms with production sets $(Y_j)_{j\in J}$, where for every $j \in J$, $Y_j \subset \mathbb{L}$. The profit made by the firm $j \in J$ is distributed among the consumers following a share function $\theta_j : A \to \mathbb{R}_+$. The share functions are supposed to be integrable and to satisfy for each $j \in J$, $\int_A \theta_j(a) d\mu(a) = 1$. If we let for each $a \in A$,

$$Y(a) := \sum_{j\in J} \theta_j(a) \overline{\mathrm{co}}\, Y_j$$

then we define an economy $\mathcal{E}' := ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, Y, P, e))$. If the production sector of the private ownership economy satisfies $\sum_{j\in J} Y_j$ is closed convex, then for all $p \in \mathbb{L}'$ and for almost every $a \in A$,

$$\int_A Y(a) d\mu(a) = \sum_{j\in J} Y_j \quad \text{and} \quad \sup p\,(Y(a)) = \sum_{j\in J} \theta_j(a) \sup p\,(Y_j)\,.$$

It follows that the notion (defined in Hildenbrand [15]) of Walrasian equilibrium for the private ownership economy $\mathcal{E}$, and the notion (defined in this paper) of Walrasian equilibrium for the associated economy $\mathcal{E}'$, coincide.

In a *coalition production economy* $\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, P, e), \mathbf{Y})$, the production sector is defined for every coalition $E \in \mathcal{A}$ by a production set $\mathbf{Y}(E) \subset \mathbb{L}$. In the framework of Hildenbrand [15], the correspondence $\mathbf{Y} : \mathcal{A} \twoheadrightarrow \mathbb{L}$ is supposed to be countably additive and to admit a Radon-Nikodym derivative. If we let $Y : A \twoheadrightarrow \mathbb{L}$ be a Radon-Nikodym derivative of $\mathbf{Y}$ then we define an economy $\mathcal{E}' = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, Y, P, e))$. If $\mathbf{Y}(A)$ is closed convex, then for every $p \in \mathbb{L}'$ and for every coalition $E \in \mathcal{A}$,

$$\sup p\,(\mathbf{Y}(E)) = \int_E \sup p\,(Y(a))\, d\mu(a).$$

Hence the notion (defined in Hildenbrand [15]) of Walrasian equilibrium for the coalition economy $\mathcal{E}$, and the notion (defined in this paper) of Walrasian equilibrium for the associated economy $\mathcal{E}'$, coincide.

3.3.3 The Assumptions

We present the list of assumptions that the economy $\mathcal{E}$ will be required to satisfy. On the consumption side we consider both non-ordered but convex preferences (Assumption C_n) and partially ordered (possibly incomplete) but non-convex preferences (Assumption C_p).

Assumption ($\mathbf{C}_n$). ***[non-ordered but convex]*** *For almost every agent $a \in A$,*

(i) *the consumption set $X(a)$ is closed convex and P_a is continuous, that is, for each bundle $x \in X(a)$, $P_a(x)$ is s-open in $X(a)$ and $P_a^{-1}(x)$ is w-open in $X(a)$,*

(ii) *the preference relation $P(a)$ is convex, that is, for each bundle $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$, and if a belongs to the non-atomic [5] part of $(A, \mathcal{A}, \mu)$, then $X(a) \setminus P_a^{-1}(x)$ is convex.*

Remark 3.3.4. When $X(a) \setminus P_a^{-1}(x)$ is supposed to be convex, the set $P_a^{-1}(x)$ is w-open in $X(a)$ if and only if it is s-open in $X(a)$.

Remark 3.3.5. Note that if $P(a)$ is partially ordered, then assuming that for all $x \in X(a)$, $X(a) \setminus P_a^{-1}(x)$ is convex, implies that for all $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$. In particular, Assumption C_n is automatically valid under Assumptions (A1-4) in Podczeck [21] and under Assumptions (3.1) and (3.2) in Khan and Yannelis [18].

Assumption ($\mathbf{C}_p$). ***[partially ordered but non-convex]*** *For a.e. $a \in A$,*

[5] An element $E \in \mathcal{A}$ is an atom of $(A, \mathcal{A}, \mu)$ if $\mu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\mu(B) = 0$ or $\mu(E \setminus B) = 0$.

(i) *the consumption set $X(a)$ is closed convex and P_a is continuous,*

(ii) *if a belongs to the non-atomic part of A then $P(a)$ is a partial order on $X(a)$, and if a belongs to an atom of A, then the relation $P(a)$ is convex, that is for each bundle $x \in X(a)$, $x \notin \operatorname{co} P_a(x)$.*

Remark 3.3.6. Following the notations of Section 3.2, when preferences are ordered, then

$$X(a) \setminus P_a^{-1}(x) = \{y \in X(a) \mid y \succeq_a x\}.$$

If $\{y \in X(a) \mid y \succeq_a x\}$ is supposed to be convex then the relation $P(a)$ is automatically convex. In particular, Assumption C_p is implied by Assumptions $\mathrm{E}(1-3)$ and $\mathrm{B}(1-2)$ in Podczeck [20], by Assumptions $\mathrm{a}(2-3)$ in Rustichini and Yannelis [22] and by Assumptions (3.1) and (3.2) in Khan and Yannelis [18]. In these three papers, preferences are supposed to be ordered, but in Assumption C_p, preferences are only required to be partially ordered.

We say that two agents a and b are equivalent, noted $a \sim b$, if $\mu(a) = \mu(b)$, $X(a) = X(b)$, $e(a) = e(b)$, $P(a) = P(b)$ and $Y(a) = Y(b)$. Two equivalent agents play the same role in the economy. The binary relation $\sim$ is an equivalence. Each equivalence class represents a *type* of consumers. We let A^{na} be the non-atomic part of A. To deal with partially ordered but non-convex preferences, we need the following assumption.

Assumption (A). *If $F : A^{na} \twoheadrightarrow \mathbb{L}$ is a graph measurable and integrably bounded correspondence with non-empty and w-compact values, such that for all $(b,c) \in A^{na}$, $b \sim c$ implies $F(b) = F(c)$, then*

$$\int_{A^{na}} \overline{\operatorname{co}}\, F(a) d\mu(a) = \int_{A^{na}} F(a) d\mu(a).$$

Remark 3.3.7. Following Theorem 3.1. in Podczeck [20], Assumption A is implied by Assumptions A1 − 2 in [20] which formulate that there are many agents of (almost) every type. If there exists a fixed w-compact set K such that for all $a \in A^{na}$, $F(a) \subset K$ then Assumption A1 (many more agents than commodities) in Rustichini and Yannelis [22] implies Assumption A. For several refinements of the Lyapunov Theorem, we refer to Tourky and Yannelis [24].

Assumption (C). *[**Consumption side**] Assumptions C_p and A are valid, or Assumption C_n is valid.*

Assumption (M). *[**Measurability**] The correspondences X and Y are graph measurable, that is,*

$$\{(a,x) \in A \times \mathbb{L} \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}) \quad \textit{and} \quad \{(a,y) \in A \times \mathbb{L} \mid y \in Y(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L})$$

and the correspondence of preferences P is lower graph measurable, that is,

$$\forall y \in S(X) \quad \{(a,x) \in A \times \mathbb{L} \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}).$$

Remark 3.3.8. Under Assumption C, the correspondence X and for all $x \in S(X)$, the correspondence $R_x : A \to \mathbb{L}$ defined by $R_x(a) = \{y \in X(a) \mid (y,x) \notin P(a)\}$ is s-closed valued. If we suppose that Y is s-closed valued, then following Propositions 3.6.2 and 3.6.6, Assumption M is valid if and only if the correspondences X and Y are measurable and for all measurable selection $x \in S(X)$, the correspondence R_x is measurable. It follows that if A is a finite set and $\mathcal{A} = 2^A$, Assumption M is then automatically valid. Moreover, under Assumption C, if preferences are ordered, following Proposition 3.6.5, we can replace in Assumption M, the lower graph measurability of P by the Aumann measurability of P, that is

$$\forall x, y \in S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

Remark 3.3.9. In Khan and Yannelis [18] and Podczeck [20], the correspondences X and P are supposed to be graph measurable. Following Proposition 3.6.4, Assumption M is then valid.

Remark 3.3.10. In Podczeck [21], it is assumed that preferences are Aumann measurable. Applying Proposition 3.6.5, in the framework of Podczeck [21], P is lower graph measurable and Assumption M is valid.

Assumption (P). *[**Production side**] The aggregate production set Y_Σ is a closed convex subset of $\mathbb{L}$.*

If we let $\tilde{Y} : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined for all $a \in A$ by

$$\tilde{Y}(a) := \mathrm{cl}\left(\overline{\mathrm{co}}\, Y(a) + A(Y_\Sigma)\right),$$

then following Proposition 3.6.7, $\tilde{Y}$ satisfies Assumption P and $\mathcal{E}$ has a free-disposal satiation quasi-equilibrium if and only if $\tilde{\mathcal{E}} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, \tilde{Y}, P, e))$ has a free-disposal satiation quasi-equilibrium.

Assumption (S). *[**Survival**] For almost every $a \in A$,*

$$0 \in \left(\{e(a)\} + \tilde{Y}(a) - X(a)\right).$$

Remark 3.3.11. Assumption S means that we have compatibility between individual needs and resources. In Khan and Yannelis [18] and Podczeck [21], the initial endowment is supposed to lie in the consumption set, that is for a.e. $a \in A$, $X(a) \cap \{e(a)\} \neq \emptyset$.

Assumption (IB). *[**Integrably Bounded**] The consumption sets correspondence X is integrably bounded with w-compact valued.*

Remark 3.3.12. We can find Assumption IB in Khan and Yannelis [18], Podczeck [20], [21] and Rustichini and Yannelis [22]. In order to apply Theorem 3.6.1, this assumption is the natural framework to deal with general Banach commodity spaces. Note that under Assumptions M, S and B, the aggregate consumption set X_Σ is non-empty.

Assumption (LNS). *[**Local Non Satiation**] For almost every agent $a \in A$, for all bundle $x \in X(a)$,*

(i) *if x is a satiation point, that is $P_a(x) = \emptyset$, then for all $y \in Y(a)$, $x \geqslant e(a) + y$;*

(ii) *if x is not a satiation point, then $x \in \overline{\mathrm{co}}\, P_a(x)$.*

Remark 3.3.13. In Podczeck [20] and [21], economies in consideration are free-disposal exchange economies, that is, for all $a \in A$, $Y(a) = -\mathbb{L}_+$. It follows that Assumptions $B4 - 5$ in [20] and $C5 - 6$ in [21] imply Assumption LNS.

Assumption (SS). *[**Strong Survival**] For almost every agent $a \in A$,*

$$\left(\{e(a)\} + \tilde{Y}(a) - X(a)\right) \cap s - \mathrm{int}\mathbb{L}_+ \neq \emptyset.$$

Remark 3.3.14. In the framework of exchange economies, Podczeck [20], [21] and Khan and Yannelis [18] supposed that for almost every agent $a \in A$, $[\{e(a)\} - X(a)] \cap s - \mathrm{int}\mathbb{L}_+ \neq \emptyset$. This obviously implies that Assumption SS is valid.

Assumption (FD). *[**Free Disposal**] The aggregate production set is free-disposal, that is, $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$.*

3.3.4 Existence result

Theorem 3.3.1. *If $\mathcal{E}$ is an economy satisfying Assumptions* ***C, M, P, S, IB*** *and* ***LNS****, then there exists a free-disposal quasi-equilibrium (x^*, y^*, p^*). If moreover $\mathcal{E}$ satisfies* ***SS****, then (x^*, y^*, p^*) is a free-disposal Walrasian equilibrium. If moreover $\mathcal{E}$ satisfies* ***SS*** *and* ***FD****, then (x^*, y^*, p^*) is a Walrasian equilibrium.*

Remark 3.3.15. In the framework of economies with convex preferences, Theorem 3.3.1 generalizes Theorem 5.1. in Podczeck [20], to economies with non-ordered preferences and with a non-trivial production sector. Under Assumption LNS, Theorem 3.3.1 generalizes the Main Theorem of Khan and Yannelis [18], to production economies with possibly non-ordered preferences. Although these authors succeed in proving the existence of a competitive equilibrium without Assumption LNS, they assume that for some s-compact subset of the commodity space, say K, the endowment of each agent belongs to K.

Remark 3.3.16. For economies with possibly non-convex preferences, Theorem 3.3.1 generalizes Theorem 5.1. in Podczeck [20], to economies with a non-trivial production sector and with possibly incomplete preferences. Under Assumption LNS, Theorem 3.3.1 generalizes Theorem 6.1. in Rustichini and Yannelis [22], to production economies. Although these authors succeed in proving the existence of a competitive equilibrium without Assumption LNS, they assume that for some s-compact subset of the commodity space, say K, the endowment of each agent belongs to K.

3.4 Discretization of measurable correspondences

3.4.1 Notations and definitions

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a separable metric space. A function $f : A \to D$ is *measurable* if for each open set $G \subset D$, $f^{-1}(G) \in \mathcal{A}$ where $f^{-1}(G) := \{a \in A \mid f(a) \in G\}$. A correspondence $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^{-}(G) := \{a \in A \mid F(a) \cap G \neq \emptyset\}$.

Definition 3.4.1. A partition $\sigma = (A_i)_{i \in I}$ of A is a *measurable partition* if for all $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of A is *subordinated to the partition* σ if there exists a family $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $A^\sigma = \{a_i \mid i \in I\}$.

3.4.1.1 Simple functions subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the application which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i)\chi_{A_i} ,$$

where χ_{A_i} is the characteristic [6] function associated with A_i. Note that the sum is well defined since there exists at most one non zero factor.

Definition 3.4.2. A function $s : A \to D$ is called *a simple function subordinated* to f if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $s = \phi(\sigma, A^\sigma)(f)$.

3.4.1.2 Simple correspondences subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the application which maps each measurable

[6] That is, for all $a \in A$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

correspondence $F : A \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i \in I} F(a_i)\chi_{A_i}.$$

Definition 3.4.3. A correspondence $S : A \to D$ is called a *simple correspondence subordinated* to a correspondence F if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $S = \psi(\sigma, A^\sigma)(F)$.

Remark 3.4.1. If f is a function from A to D, let $\{f\}$ be the correspondence from A into D, defined for all $a \in A$ by $\{f\}(a) := \{f(a)\}$. We check that

$$\psi(\sigma, A^\sigma)(F) = \{\phi(\sigma, A^\sigma)(f)\}.$$

3.4.1.3 Hyperspace

The space of all non-empty subsets of D is noted $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wisjman topology on $\mathcal{P}^*(D)$, that is the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x,.))_{x \in D}$. The Hausdorff semi-metric H_d on $\mathcal{P}^*(D)$ is defined by

$$\forall (A,B) \in \mathcal{P}^*(D) \quad H_d(A,B) := \sup\{|d(x,A) - d(x,B)| \mid x \in D\}.$$

A subset C of D is the Hausdorff limit of a sequence $(C_n)_{n \in \mathbb{N}}$ of subsets of D, if

$$\lim_{n \to \infty} H_d(C_n, C) = 0.$$

3.4.2 Approximation of measurable correspondences

Hereafter we assert that for a countable set of measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence. The proof of the following theorem is given in Martins Da Rocha [19].

Theorem 3.4.1. *Let $\mathcal{F}$ be a countable set of measurable correspondences with non-empty values from A into D and let $\mathcal{G}$ be a finite set of integrable functions from A to $\mathbb{R}$. There exists a sequence $(\sigma^n)_{n \in \mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i \in I^n}$ of A, satisfying the following properties.*

(a) *Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $F \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F. The following properties are then satisfied.*

1. *For all $a \in A$, $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_{n \in \mathbb{N}}$, i.e. ,*

$$\forall a \in A \quad \forall x \in A \quad \lim_{n \to \infty} d(x, F^n(a)) = d(x, F(a)).$$

2. *If D is d-bounded then for all $x \in D$ the real valued function $d(x, F(.))$ is the uniform limit of the sequence $(d(x, F^n(.)))_{n \in \mathbb{N}}$.*
3. *If D is d-totally bounded then F is the uniform Hausdorff limit of the sequence $(F^n)_{n \in \mathbb{N}}$.*

(b) *There exists a sequence $(A^n)_{n \in \mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $n \in \mathbb{N}$, if we let $f^n := \phi(\sigma^n, A^n)(f)$ be the simple function subordinated to each $f \in \mathcal{G}$, then*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g \in \mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n \to \infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remark 3.4.2. The property (a1) implies in particular that, if $(x^n)_{n\in\mathbb{N}}$ is a sequence of D, d-converging to $x \in D$, then

$$\forall a \in A \quad \lim_{n\to\infty} d(x^n, F^n(a)) = d(x, F(a)).$$

It follows that if F is non-empty closed valued, then property (a1) implies that

$$\forall a \in A \quad \operatorname{ls} F^n(a) \subset F(a).$$

3.5 Proof of the existence theorem

3.5.1 Free-disposal satiation equilibria

Hereafter, we introduce an auxiliary concept of quasi-equilibrium for an economy $\mathcal{E}$.

Definition 3.5.1. An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ is a *free-disposal satiation quasi-equilibrium* of the economy $\mathcal{E}$ if $p^* > 0$ and if the following properties are satisfied.

(i) For almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*(x) \geqslant p^*(y) + p^*(e(a)).$$

(ii)

$$\int_A x^*(a) d\mu(a) \leqslant \int_A e(a) d\mu(a) + \int_A y^*(a) d\mu(a).$$

If (x^*, y^*, p^*) is a free-disposal quasi-equilibrium of an economy $\mathcal{E}$, then (x^*, y^*, p^*) is clearly a free-disposal satiation quasi-equilibrium of $\mathcal{E}$.

Remark 3.5.1. Under Assumption LNS, every free-disposal satiation quasi-equilibrium (x^*, y^*, p^*) of an economy $\mathcal{E}$, is in fact a free-disposal quasi-equilibrium of $\mathcal{E}$.

Following Remarks 3.3.1, 3.3.2 and 3.5.1, to prove the existence of a Walrasian equilibrium, it is sufficient (under Assumptions SS, LNS and FD) to prove the following lemma.

Lemma 3.5.1. *If $\mathcal{E}$ is an economy satisfying Assumptions **C**, **M**, **P**, **S** and **IB**, then a free-disposal satiation quasi-equilibrium of $\mathcal{E}$ exists.*

3.5.2 Existence of free-disposal satiation equilibria for *polytope* economies

We propose first to prove an auxiliary existence result (the following Lemma 3.5.2) for *polytope* economies, that is, economies satisfying the following assumption K. This first step allows us to isolate the crucial aspect of the new approach, which is the approximation of economies with a measure space of agents (measurable correspondences) by a sequence of economies with a finite set of agents (resp. simple correspondences). Moreover, the framework of *polytope* economies allows us to deal with non-ordered but convex preferences, as well as, ordered but non-convex preferences.

Assumption (K). *There exist a finite set $K = \{0, \cdots, r\}$ and Bochner integrable functions $(x_k)_{k\in K}$, $(y_k)_{k\in K}$ from A to $\mathbb{L}$ such that for almost every agent $a \in A$,*

$$X(a) = \operatorname{co}\{x_0(a), \cdots, x_r(a)\} \quad \textit{and} \quad Y(a) = \operatorname{co}\{y_0(a), \cdots, y_r(a)\}.$$

Lemma 3.5.2. *If $\mathcal{E}$ is an economy satisfying Assumptions C, M, P, S, and K, then a free-disposal satiation quasi-equilibrium of $\mathcal{E}$ exists.*

Proof. We can suppose (considering a translation if necessary) that for almost every $a \in A$, $e(a) = 0$. Following Proposition 3.6.2, the correspondences X and Y are measurable. Then following Proposition

3.6.1, there exist a sequence $(f_k)_{k\in\mathbb{N}}$ of measurable selections of X and a sequence $(g_k)_{k\in\mathbb{N}}$ of measurable selections of Y such that for all $a \in A$,

$$X(a) = s\text{-cl}\,\{f_k(a) \mid k \in \mathbb{N}\} \quad \text{and} \quad Y(a) = s\text{-cl}\,\{g_k(a) \mid k \in \mathbb{N}\}.$$

Following Assumption S, we can suppose without any loss of generality that for every $a \in A$, $x_0(a) = f_0(a) = g_0(a) = y_0(a)$. We let for all $k \in \mathbb{N}$, $R_k : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined by $R_k(a) := \{x \in X(a) \mid f_k(a) \notin P_a(x)\}$. Then for almost every agent $a \in A$, for all $x \in \mathbb{L}$,

$$d(x, X(a)) = 0 \Leftrightarrow x \in X(a) \quad \text{and} \quad d(x, Y(a)) = 0 \Leftrightarrow x \in Y(a),$$

and for all $x \in X(a)$,

$$\forall k \in \mathbb{N} \quad d(x, R_k(a)) > 0 \Leftrightarrow f_k(a) \in P_a(x).$$

Following Assumption K, we let for each $a \in A$,

$$h(a) := \max\{\|x_k(a)\|, \|y_k(a)\| \mid 0 \leqslant k \leqslant r\}.$$

It follows that the correspondences X and Y are integrably bounded by h. Applying [7] Theorem 3.4.1, there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of measurable partitions $\sigma^n = (A_i^n)_{i\in S^n}$ of $(A, \mathcal{A})$, and a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets $A^n = \{a_i^n \mid i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying the following properties.

Fact 3.5.1. For all $a \in A$,

(i) for all $n \in \mathbb{N}$, $h^n(a) \leqslant 1 + h(a)$ and for all $(k, j) \in \mathbb{N} \times K$,

$$\forall k \in \mathbb{N} \quad s\text{-}\lim_n (f_k^n(a), g_k^n(a)) = (f_k(a), g_k(a))$$

and

$$\forall j \in K \quad s\text{-}\lim_n (x_j^n(a), y_j^n(a)) = (x_j(a), y_j(a))\ ;$$

(ii) for all sequence $(x^n)_{n\in\mathbb{N}}$ of $\mathbb{L}$, s-converging to $x \in \mathbb{L}$,

$$\lim_{n\to\infty} d(x^n, X^n(a)) = d(x, X(a))\,, \quad \lim_{n\to\infty} d(x^n, Y^n(a)) = d(x, Y(a))$$

and

$$\lim_{n\to\infty} d(x^n, R_k^n(a)) = d(x, R_k(a)),$$

where d is the distance function associated to the norm $\|.\|$.

We let, for each $a \in A$,

$$K_1(a) := \overline{\text{co}} \bigcup_{k\in K} \{x_k^n(a) \mid n \in \mathbb{N}\} \quad \text{and} \quad K_2(a) := \overline{\text{co}} \bigcup_{k\in K} \{y_k^n(a) \mid n \in \mathbb{N}\}.$$

A direct consequence of Fact 3.5.1 together with Theorem 5.20 in Aliprantis and Border [1] and Theorem 8.2.2 in Aubin and Frankowska [2], is the following result.

Fact 3.5.2. The correspondences K_1 and K_2 are measurable, integrably bounded with non-empty, s-compact and convex values.

We construct now a sequence of economies with a finite set of consumers. We distinguish two cases. In the first case (Claim 3.5.1) preferences are possibly non-ordered but convex, in the second case (Claim 3.5.2) preferences are ordered but possibly non-convex.

Claim 3.5.1. If $\mathcal{E}$ satisfies Assumptions C_n, then there exists a free-disposal satiation quasi-equilibrium.

[7] Note that for each $k \in \mathbb{N}$ the correspondence R_k is graph measurable and with closed values. Following Proposition 3.6.2, it is then measurable.

Proof. For all $n \in \mathbb{N}$, we note $\mathcal{G}^n$ the following *finite* production economy

$$\mathcal{G}^n = \left(\langle \mathbb{L}', \mathbb{L} \rangle , (X_i^n, Y_i^n - \mathbb{L}_+, P_i^n)_{i \in I^n} \right)$$

where $I^n := \{ i \in S^n \mid \mu(A_i^n) \neq 0 \}$ is the finite set of consumers. The consumption set of consumer $i \in I^n$ is given by $X_i^n := \mu(A_i^n) X(a_i^n)$ [8] and the production set is given by $Y_i^n - \mathbb{L}_+$, where $Y_i^n := \mu(A_i^n) Y(a_i^n)$. The preferences are given by $P_i^n := \mu(A_i^n) P(a_i^n)$.

We assert that the economy $\mathcal{G}^n$ satisfies all the assumptions [9] of Proposition 4 in Florenzano [14] and thus there exists $(x_i^n)_{i \in I^n} \in \prod_{i \in I^n} X_i^n$ such that $0 \notin G$ where [10]

$$G := \mathbb{Q} - \operatorname{co} \bigcup_{i \in I^n} \left(\operatorname{co} P_i^n(x_i^n) - \operatorname{co} Y_i^n - \mathbb{L}_+ \right).$$

Applying Proposition 3.6.8 there exists $(y_i^n)_{i \in I^n} \in \prod_{i \in I^n} Y_i^n$ and $p^n \in \mathbb{L}' \setminus \{0\}$ satisfying $p^n > 0$, $\sum_{i \in I^n} x_i^n \leqslant \sum_{i \in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n (x - y) \geqslant 0$.

Let, for all $n \in \mathbb{N}$,

$$x^n := \sum_{i \in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i \in I^n} \frac{y_i^n}{\mu(A_i^n)} \chi_{A_i^n}.$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a) d\mu(a) \leqslant \int_A y^n(a) d\mu(a) \tag{3.1}$$

$$\forall a \in \bigcup_{i \in I^n} A_i^n \quad (x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n(x) \geqslant p^n(y). \tag{3.2}$$

Note that for almost every $a \in A$, for each $n \in \mathbb{N}$, $x^n(a) \in K_1(a)$ and $y^n(a) \in K_2(a)$. Applying Fact 3.5.2 and Theorem 3.6.1 [11], there exists Bochner integrable functions $x^*, y^* : A \to \mathbb{L}$ such that

$$\int_A x^* d\mu = \lim_{n \to \infty} \int_A x^n d\mu \quad \text{and} \quad \int_A y^* d\mu = \lim_{n \to \infty} \int_A y^n d\mu \tag{3.3}$$

$$\text{for a.e. } a \in A^{na} \quad x^*(a) \in \overline{\operatorname{co}}\, s\text{-}\mathrm{ls}\, \{x^n(a)\} \quad \text{and} \quad y^*(a) \in \overline{\operatorname{co}}\, s\text{-}\mathrm{ls}\, \{y^n(a)\} \tag{3.4}$$

$$\text{for all } a \in A^{pa} \quad x^*(a) \in s\text{-}\mathrm{ls}\, \{x^n(a)\} \quad \text{and} \quad y^*(a) \in s\text{-}\mathrm{ls}\, \{y^n(a)\} \tag{3.5}$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ et A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$. Since, for all $n \in \mathbb{N}$, $p^n \in \mathbb{L}'_+ \setminus \{0\}$, we may suppose (extracting a subsequence if necessary) that $(p^n)_{n \in \mathbb{N}}$ w^*-converging to p^*, with $p^* \in \mathbb{L}'_+ \setminus \{0\}$.

We propose to prove that (x^*, y^*, p^*) is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. We let

$$A_0 := \bigcup_{n \in \mathbb{N}} \bigcup_{i \in S^n \setminus I^n} A_i^n,$$

then we easily check that $\mu(A_0) = 0$. Let now A' be a measurable subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost every where* assumptions and properties are satisfied for all $a \in A'$.

To prove condition (ii) of Definition 3.5.1, we need to prove that $(x^*, y^*) \in S^1(X) \times S^1(Y)$. Let $a \in A'$, by construction, we have that for every $n \in \mathbb{N}$, $x^n(a) \in X^n(a)$, and thus, for every $n \in \mathbb{N}$, $d(x^n(a), X^n(a)) = 0$. We apply Fact 3.5.1 to conclude that for all $\xi \in s\text{-}\mathrm{ls}\,\{x^n(a)\}$, $d(\xi, X(a)) = 0$. It follows that $s\text{-}\mathrm{ls}\,\{x^n(a)\} \subset X(a)$. Since $x^*(a) \in \overline{\operatorname{co}}\, s\text{-}\mathrm{ls}\,\{x^n(a)\}$, applying Assumption K, we get that $x^*(a) \in X(a)$. We prove similarly that $y^* \in S^1(Y)$. Following (3.1) and (3.3), condition (ii) is thus valid.

[8] The consumer a_i^n "represents" the coalition A_i^n.
[9] In particular the Survival Assumption is valid, since for almost every $a \in A$, $f_0(a) = g_0(a)$.
[10] We refer to Proposition 3.6.8 for the definition of the $\mathbb{Q}$-convex hull.
[11] Since the correspondences K_1 and K_2 have s-compact values, we have that w-ls $= s$-ls.

We will now prove that (x^*, y^*, p^*) satisfies condition (i) of Definition 3.5.1. Let $a \in A'$ and $(x, y) \in P_a(x^*(a)) \times Y(a)$. We let $\mathcal{I}$ be the set of strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_{k\in\mathbb{N}}$ s-converges to x and that $(g_{\psi(k)}(a))_{k\in\mathbb{N}}$ s-converges to y. To prove that $p^*(x-y) \geqslant 0$, it is sufficient to prove that for all k large enough, $p^*\left(f_{\phi(k)}(a)\right) \geqslant p^*\left(g_{\psi(k)}(a)\right)$. Following Assumption C_n, there exist $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(x^*(a))$. Let $k \geqslant k_0$, we let $i := \phi(k)$ and $j := \psi(k)$.

We assert that there exists $\alpha \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N} \quad \left(f_i^{\alpha(n)}(a), g_j^{\alpha(n)}(a)\right) \in P_a^{\alpha(n)}\left(x^{\alpha(n)}(a)\right) \times Y^{\alpha(n)}(a). \tag{3.6}$$

Indeed, by definition of $Y^n(a)$, we have that $g_j^n(a) \in Y^n(a)$. Suppose now that for all $\alpha \in \mathcal{I}$, there exist $\beta \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N} \quad d\left(x^{\alpha\circ\beta(n)}(a), R_i^{\alpha\circ\beta(n)}(a)\right) = 0.$$

Applying (ii) of Fact 3.5.1, it follows that for all $\xi \in s\text{-ls}\,\{x^n(a)\}$, $d(\xi, R_i(a)) = 0$, that is, $\xi \in R_i(a)$. But with Assumption C_n we have that $R_i(a)$ is closed convex, if a belongs to the non-atomic part of $(A, \mathcal{A}, \mu)$. Applying (3.4) and (3.5), we conclude that $x^*(a) \in R_i(a)$, that is, $f_i(a) \notin P_a(x^*(a))$. Contradiction.

Applying (3.6) together with (3.2), we obtain that,

$$\forall n \in \mathbb{N} \quad p^{\alpha(n)}\left(f_i^{\alpha(n)}(a) - g_j^{\alpha(n)}(a)\right) \geqslant 0.$$

Applying Fact 3.5.1, we have that $(f_i^n(a) - g_j^n(a))_{n\in\mathbb{N}}$ s-converges to $f_i(a) - g_j(a)$. Since $(p^n)_{n\in\mathbb{N}}$ w^*-converges to p^*, we get that $p^*(f_i(a)) \geqslant p^*(g_j(a))$. $\square$

We consider now the case of ordered but possibly non-convex preferences.

Claim 3.5.2. If $\mathcal{E}$ satisfies Assumptions C_p and A, then a free-disposal satiation quasi-equilibrium exists.

Proof. Following Theorem 2 in Sondermann [23], for almost every $a \in A$, there exists an upper s-semi-continuous utility function u_a representing the binary relation $P(a)$ on $X(a)$, in the sense that

$$(x, x') \in P(a) \Longrightarrow u_a(x) < u_a(x').$$

We note $A^{na} \subset A$ the non-atomic part of $(A, \mathcal{A}, \mu)$. We let, for almost every $a \in A^{na}$,

$$\tilde{P}(a) := \{(x, x') \in X(a) \times X(a) \mid u_a(x) < u_a(x')\}$$

and for each $a \in A^{pa}$, $\tilde{P}(a) := P(a)$. Note that for almost every $a \in A$, $P(a) \subset \tilde{P}(a)$. We define the correspondence $\tilde{R}$ from A into $\mathbb{L} \times \mathbb{L}$ by, for almost every $a \in A^{na}$, $\tilde{R}(a) := \{(z, z') \in X(a) \times X(a) \mid u_a(z) \leqslant u_a(z')\}$; and for all $a \in A^{pa}$, $\tilde{R}(a) := R(a)$.

In order to use the same *limit* argument as Claim 3.5.1, we define *convex* preferences. This construction is borrowed from Hildenbrand [16]. We define $\hat{P} : A \to \mathbb{L} \times \mathbb{L}$ by, for all a in the non-atomic part A^{na} of $(A, \mathcal{A}, \mu)$,

$$\hat{P}(a) := \{(x, x') \in X(a) \times X(a) \mid x \notin \overline{\mathrm{co}}\,\tilde{R}_a(x')\}$$

and for all a in the purely atomic part A^{pa}, $\hat{P}(a) = P(a)$. For almost every $a \in A$, for each $y \in X(a)$, $\hat{P}_a^{-1}(y)$ is s-open in $X(a)$. Moreover, the binary relation $\tilde{R}(a)$ is a complete pre-order on the non-atomic part of A. We check then, that for almost every $a \in A$, $\hat{P}(a)$ satisfies the following convex properties,

$$\forall x \in X(a) \quad x \notin \mathrm{co}\,\hat{P}_a(x) \quad \text{and} \quad a \in A^{na} \Rightarrow \quad X(a) \setminus \hat{P}_a^{-1}(x) \text{ is convex.}$$

We are now ready to construct the sequence of economies with a finite set of consumers. For all $n \in \mathbb{N}$, we note $\mathcal{E}^n$ the following *finite* economy $\mathcal{E}^n = (\langle \mathbb{L}', \mathbb{L}\rangle, (X_i^n, Y_i^n, P_i^n)_{i\in I^n})$ where $I^n := \{i \in S^n \mid \mu(A_i^n) \neq 0\}$ is the finite set of consumers. The consumption set of the consumer $i \in I^n$ is given by $X_i^n := \mu(A_i^n)X(a_i^n)$ and the production set is given by $Y_i^n - \mathbb{L}_+$, where $Y_i^n := \mu(A_i^n)\left[Y(a_i^n) + (1/n)\{u\}\right]$, and u is a vector in $s\text{-int}\mathbb{L}_+$. The preferences are given by $P_i^n := \mu(A_i^n)\hat{P}(a_i^n)$.

We assert that the economy $\mathcal{E}^n$ satisfies all the assumptions [12] of Proposition 4 in Florenzano [14]. It follows that there exists $(x_i^n)_{i\in I^n} \in \prod_{i\in I^n} X_i^n$ such that $0 \notin G$ where [13]

$$G := \mathbb{Q} - \mathrm{co} \bigcup_{i\in I^n} (\mathrm{co}\, P_i^n(x_i^n) - \mathrm{co}\, Y_i^n - \mathbb{L}_+) .$$

Applying Proposition 3.6.8 [14] there exists $(y_i^n)_{i\in I^n} \in \prod_{i\in I^n} Y_i^n$ and $p^n \in \mathbb{L}' \setminus \{0\}$ satisfying $p^n > 0$, $\sum_{i\in I^n} x_i^n \leqslant \sum_{i\in I^n} y_i^n$ and for all $i \in I^n$, if $(x, y) \in P_i^n(x_i^n) \times Y_i^n$ then $p^n (x - y) \geqslant 0$.

We let for all $n \in \mathbb{N}$,

$$x^n := \sum_{i\in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y_n := \sum_{i\in I^n} \left(\frac{y_i^n}{\mu(A_i^n)} - \frac{1}{n} u \right) \chi_{A_i^n} .$$

For each $n \in \mathbb{N}$, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a) d\mu(a) \leqslant \int_A y^n(a) d\mu(a) + (1/n)u \tag{3.7}$$

$$\forall a \in \bigcup_{i\in I^n} A_i^n \quad (x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n (x) > p^n (y) . \tag{3.8}$$

Since, for all $n \in \mathbb{N}$, $p^n \geqslant 0$, there exists a subsequence of $(p^n)_{n\in\mathbb{N}}$ converging to p^*, with $p^* (u) = 1$. For all $a \in A$, we let

$$B(a) = \{x \in X(a) \mid p^* (x) \leqslant \sup p^* (Y(a))\}$$

and

$$\beta(a) = \{x \in X(a) \mid p^* (x) < \sup p^* (Y(a))\}.$$

We define the correspondences D, G and H by, for all $a \in A$,

$$D(a) := \{x \in B(a) \mid P_a(x) \cap B(a) = \emptyset\}, \quad G(a) := \{x \in X(a) \mid P_a(x) \cap B(a) = \emptyset\}$$

and

$$H(a) := \{x \in X(a) \mid P_a(x) \cap \beta(a) = \emptyset\}.$$

When replacing P by $\hat{P}$, we define $\hat{G}$. Moreover, for each $n \in \mathbb{N}$, when replacing X by X^n, P by P^n, Y by Y^n and p^* by p^n, we define $B^n(a)$, $\beta^n(a)$, $D^n(a)$ and $G^n(a)$. Similarly when replacing P^n by $\tilde{P}^n$, we define $\tilde{D}^n$ and $\tilde{G}^n$. We define $\hat{G}^n$ when P^n by $\hat{P}^n$. For all $n \in \mathbb{N}$, for all $a \in A^{pa}$, $\hat{G}^n(a) = \tilde{G}^n(a) = G^n(a)$. We assert that for all $n \in \mathbb{N}$,

$$\forall a \in A^{na} \quad \hat{G}^n(a) \subset \mathrm{co}\, [\tilde{G}^n(a)] \subset \mathrm{co}\, [G^n(a)]. \tag{3.9}$$

Indeed, if $a \in A^{pa}$ then $\hat{P}^n(a) = P^n(a)$ and the result follows. Now let $a \in A^{na}$ and $x \in \hat{G}^n(a)$. The set $X^n(a)$ is s-compact, the strict-preference relation $\tilde{P}^n(a)$ is irreflexive, transitive with s-open lower sections. Hence, following a classical maximal argument, the set $\tilde{D}^n(a)$ is non-empty. Let $\tilde{x} \in \tilde{D}^n(a)$, then $\tilde{x} \in B^n(a)$, and since $x \in \hat{G}^n(a)$, we have that $(x, \tilde{x}) \notin \hat{P}^n(a)$, that is, $x \in \mathrm{co}\, \tilde{R}_a^n(\tilde{x})$. Since $\tilde{R}^n(a)$ is transitive and complete, it is straightforward to verify that $\tilde{R}_a^n(\tilde{x}) \subset \tilde{G}^n(a) \subset G^n(a)$, and thus $x \in \mathrm{co}\, [G^n(a)]$.

Since (x^n, p^n) satisfies (3.8), it follows [15] that for a.e. $a \in A$, $x^n(a) \in \hat{G}^n(a) \subset \mathrm{co}\, G^n(a)$. Note that for almost every $a \in A$, for each $n \in \mathbb{N}$, $x^n(a) \in K_1(a)$ and $y^n(a) \in K_2(a)$. Applying Fact 3.5.2 and Theorem 3.6.1, there exists Bochner integrable functions $x^*, y^* : A \to \mathbb{L}$ such that

$$\int_A (x^*(a), y^*(a)) d\mu(a) = \lim_{n\to\infty} \int_A (x^n(a), y^n(a)) d\mu(a) \tag{3.10}$$

[12] In particular the Survival Assumption is valid, since for almost every $a \in A$, $f_0(a) = g_0(a)$.
[13] We refer to Proposition 3.6.8 for the definition of the $\mathbb{Q}$-convex hull.
[14] Contrary to the context of claim 3.5.1 we do not now if P_i^n has s-open values, and thus we do not know if $0 \notin \mathrm{co}\, G$.
[15] This is the reason why we introduce u in the construction of Y_i^n.

$$\text{for a.e. } a \in A^{na} \quad x^*(a) \in \overline{\text{co}}\, s\text{-ls}\,\{x^n(a)\} \quad \text{and} \quad y^*(a) \in \overline{\text{co}}\, s\text{-ls}\,\{y^n(a)\} \tag{3.11}$$

$$\text{for all } a \in A^{pa} \quad x^*(a) \in s\text{-ls}\,\{x^n(a)\} \quad \text{and} \quad y^*(a) \in s\text{-ls}\,\{y^n(a)\}. \tag{3.12}$$

Following verbatim the arguments of Claim 3.5.1,

$$s\text{-ls}\, X^n(a) \subset X(a) \quad \text{and} \quad s\text{-ls}\, Y^n(a) \subset Y(a).$$

With Assumption K, $\overline{\text{co}}\, X(a) = X(a)$ and $\overline{\text{co}}\, Y(a) = Y(a)$, it follows that $x^* \in S^1(X)$ and $y^* \in S^1(Y)$. Applying (3.7),

$$\int_A x^*(a) d\mu(a) \leqslant \int_A y^*(a) d\mu(a). \tag{3.13}$$

Once again, following verbatim the arguments of Claim 3.5.1, we prove that for almost every $a \in A$,

$$s\text{-ls}\,[H^n(a)] \subset H(a).$$

Applying Carathéodory Convexity Theorem, for almost every $a \in A$,

$$s\text{-ls}\,(\text{co}\,[H^n(a)]) \subset \text{co}\, s\text{-ls}\,(H^n(a)) \subset \text{co}\, H(a).$$

It follows [16] that for almost every $a \in A$,

$$a \in A^{na} \Rightarrow x^*(a) \in \overline{\text{co}}\, H(a) \quad \text{and} \quad a \in A^{pa} \Rightarrow x^*(a) \in H(a).$$

We assert that the correspondence H is graph measurable. Indeed, we let $A^\beta := \{a \in A \mid \beta(a) \neq \emptyset\}$. Since X and Y are graph measurable, then β is graph measurable and $A^\beta \in \mathcal{A}$. Applying Proposition 3.6.3, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of measurable selections of $\beta_{|A^\beta}$ satisfying, for all $a \in A^\beta$, $(h_n(a))_{n\in\mathbb{N}}$ is dense in $\beta(a)$. We let, for each $n \in \mathbb{N}$, $z_n(a) := h_n(a)$ if $a \in A^\beta$ and $z_n(a) := f_n(a)$ elsewhere. It follows that

$$\forall a \in A^\beta \quad H(a) = \bigcap_{n\in\mathbb{N}} R_{z_n}(a) \quad \text{and} \quad \forall a \in A \setminus A^\beta \quad H(a) = X(a),$$

where $R_{z_n}(a) = \{x \in X(a) \mid (x, z_n) \notin P(a)\}$. Applying Assumption M, for each $n \in \mathbb{N}$, R_{z_n} is graph measurable and H is then graph measurable.

We apply now Assumption A,

$$\int_A x^*(a) d\mu(a) \in \int_{A^{na}} \overline{\text{co}}\,[H(a)] d\mu(a) + \int_{A^{pa}} H(a) d\mu(a) = \int_A H(a) d\mu(a).$$

That is, there exists $\bar{x} \in S^1(X)$ such that for almost every agent $a \in A$, $\bar{x}(a) \in H(a)$ and following (3.13), $\int_A \bar{x} \leqslant \int_A e(a)d\mu(a) + \int_A y^*(a)d\mu(a)$. It follows that $(\bar{x}, y^*, p^*)$ is a free-disposal satiation quasi-equilibrium of the economy $\mathcal{E}$. □

The proof of Lemma 3.5.2 is a direct consequence of Claim 3.5.1 and Claim 3.5.2. □

3.5.3 Proof of Lemma 3.5.1

We now apply Lemma 3.5.2 to prove Lemma 3.5.1.

Proof. Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S and IB. Following Proposition 3.6.7, we can suppose without any loss of generality that for almost every $a \in A$, $Y(a) = \tilde{Y}(a)$ is a closed convex subset of $\mathbb{L}$ and that for almost every $a \in A$, $e(a) = 0$. Applying Proposition 3.6.2, the correspondences X and Y are measurable. Applying Proposition 3.6.1 together with Assumption S, there exist $f_0 \in S^1(X)$ and $g_0 \in S^1(Y)$ such that for almost every $a \in A$, $f_0(a) = g_0(a)$. Once again applying Proposition 3.6.1, there exist a sequence $(f_k)_{k\in\mathbb{N}}$ of measurable selections of X and a sequence $(g_k)_{k\in\mathbb{N}}$ of measurable selections of Y such that for all $a \in A$,

$$X(a) = s\text{-cl}\,\{f_k(a) \mid k \in \mathbb{N}\} \quad \text{and} \quad Y(a) = s\text{-cl}\,\{g_k(a) \mid k \in \mathbb{N}\}.$$

[16] Recall that for all $n \in \mathbb{N}$, $x^n(a) \in \text{co}\, G^n(a) \subset \text{co}\, H^n(a)$.

For each $n \in \mathbb{N}$, let $\mathcal{E}^n = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X^n, Y^n, P^n))$, where for each agent $a \in A$, the consumption and production sets are defined by

$$X^n(a) := \mathrm{co}\,\{f_0(a), \cdots, f_n(a)\} \subset X(a)$$

and

$$Y^n(a) := \mathrm{co}\,\{g_0(a), g_1^n(a), \cdots, g_n^n(a)\} \subset Y(a),$$

where for each $1 \leqslant k \leqslant n$, $g_k^n(a) = g_k(a)$ if $\|g_k(a)\| \leqslant n$ and $g_k^n(a) = g_0(a)$ either. The preferences are defined by $P^n(a) := P(a) \cap (X^n(a) \times X^n(a))$. For each $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies Assumptions C, M, P, S and K. Applying Lemma 3.5.2, we obtain the following fact.

Fact 3.5.3. For each $n \in \mathbb{N}$, there exists [17] $(x^n, p^n) \in S^1(X^n) \times \mathbb{L}'_+$ with $p^n \neq 0$ and such that there exists $A^n \in \mathcal{A}$, with $\mu(A \setminus A^n) = 0$ and satisfying the following properties.

(i) For every $a \in A^n$, $(x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Longrightarrow p^n\,(x - y) \geqslant 0$.

(ii) $\int_A x^n(a) d\mu(a) \in Y_\Sigma - \mathbb{L}_+$.

Applying Theorem 3.6.1, there exists a Bochner integrable function $x^* \in S^1(X)$ such that

$$\int_A x^*(a) d\mu(a) = \lim_{n \to \infty} \int_A x^n(a) d\mu(a). \tag{3.14}$$

$$\text{for a.e. } a \in A^{na} \quad x^*(a) \in \overline{\mathrm{co}}\, w\text{-ls}\,\{x^n(a)\} \tag{3.15}$$

$$\text{for all } a \in A^{pa} \quad x^*(a) \in w\text{-ls}\,\{x^n(a)\}. \tag{3.16}$$

For all $n \in \mathbb{N}$, $p^n > 0$. Since s-$\mathrm{int}\mathbb{L}_+ \neq \emptyset$, we may suppose (extracting a subsequence if necessary) that $(p^n)_{n \in \mathbb{N}}$ w^*-converges to p^*, with $p^* > 0$. Following (3.14) and (ii) of Fact 3.5.3, there exists $y^* \in S^1(Y)$ such that

$$\int_A x^*(a) d\mu(a) \leqslant \int_A y^*(a) d\mu(a). \tag{3.17}$$

For the rest of the proof, we distinguish two cases. In the first case (Claim 3.5.3) preferences are possibly non-ordered but convex, in the second case (Claim 3.5.4) preferences are ordered but possibly non-convex.

Claim 3.5.3. If $\mathcal{E}$ satisfies Assumptions C_n, then there exists a free-disposal satiation quasi-equilibrium.

Proof. We propose to prove that (x^*, y^*, p^*) is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. Following (3.17) it suffices to prove that for almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Longrightarrow p^*\,(x) \geqslant p^*\,(y)\,.$$

Let $a \in A \backslash (\cup_{n \in \mathbb{N}} A^n)$ and let $(x, y) \in P_a(x^*(a)) \times Y(a)$. We let $\mathcal{I}$ be the set of strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_{k \in \mathbb{N}}$ s-converges to x and that $(g_{\psi(k)}(a))_{k \in \mathbb{N}}$ s-converges to y. Moreover, we can suppose that for all k large enough,

$$g_{\psi(k)}(a) = g_{\psi(k)}^{\psi(k)}(a) \in Y^k(a).$$

To prove that $p^*\,(x - y) \geqslant 0$, it is sufficient to prove that for all k large enough,

$$p^*\left(f_{\phi(k)}(a)\right) \geqslant p^*\left(g_{\psi(k)}(a)\right).$$

Following Assumption C_n, there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(x^*(a))$. Let $k \geqslant k_0$, we let $i := \phi(k)$ and $j := \psi(k)$.

[17] Recall that for all $n \in \mathbb{N}$, $S^1(X^n) \subset S^1(X)$.

We can suppose that there exists $\alpha \in \mathcal{I}$ such that for all $n \in \mathbb{N}$, $(f_i(a), g_j(a)) \in P_a^{\alpha(n)}(x^{\alpha(n)}(a)) \times Y^{\alpha(n)}(a)$. Indeed, for all $n \geqslant k$, $(f_i(a), g_j(a)) \in X^n(a) \times Y^n(a)$. Suppose that for all $\alpha \in \mathcal{I}$, there exists $\beta \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N} \quad x^{\alpha \circ \beta(n)}(a) \in R_i(a).$$

Applying Assumption C_n, it follows that $w\text{-ls}\,\{x^n(a)\} \subset R_i(a)$. But $R_i(a)$ is closed convex if $a \in A^{na}$. Applying (3.15) and (3.16), we conclude that $x^*(a) \in R_i(a)$, that is, $f_i(a) \notin P_a(x^*(a))$. Contradiction. It follows that there exists $\alpha \in \mathcal{I}$ such that for all $n \in \mathbb{N}$, $(f_i(a), g_j(a)) \in P_a^{\alpha(n)}(x^{\alpha(n)}(a)) \times Y^{\alpha(n)}(a)$.

Thus applying (i) of Fact 3.5.3, we obtain that, for all $n \in \mathbb{N}$, $p^{\alpha(n)}\left(f_i(a) - g_j(a)\right) \geqslant 0$. Since $(p^n)_{n \in \mathbb{N}}$ w^*-converges to p^*, it follows that $p^*\left(f_i(a)\right) \geqslant p^*\left(g_j(a)\right)$. □

We consider now the case of ordered but possibly non-convex preferences.

Claim 3.5.4. If $\mathcal{E}$ satisfies Assumptions C_p and A, then a free-disposal satiation quasi-equilibrium exists.

Proof. Following Fact 3.5.3 and notations introduced in the proof of Lemma 3.5.2, for almost every $a \in A$,

$$\forall n \in \mathbb{N} \quad x^n(a) \in H^n(a).$$

Claim 3.5.5. We assert that for every $a \in A \setminus (\cup_{n \in \mathbb{N}} A^n)$,

$$w\text{-ls}\, H^n(a) \subset H(a).$$

Proof. Indeed, let $a \in A \setminus (\cup_{n \in \mathbb{N}} A^n)$ and $z^*(a) \in w\text{-ls}\, H^n(a)$. Since $X(a)$ is w-closed, $z^*(a) \in w\text{-ls}\, X^n(a) \subset X(a)$. To prove that $z^*(a) \in H(a)$, it is sufficient to prove that

$$(z, y) \in P_a(z^*(a)) \times Y(a) \Longrightarrow p^*\left(z\right) \geqslant p^*\left(y\right).$$

We let $\mathcal{I}$ be the set of strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_{k \in \mathbb{N}}$ s-converges to z and that $(g_{\psi(k)}(a))_{k \in \mathbb{N}}$ s-converges to y. Moreover, we can suppose that for all k large enough,

$$g_{\psi(k)}(a) = g_{\psi(k)}^{\psi(k)}(a) \in Y^k(a).$$

To prove that $p^*\left(z - y\right) \geqslant 0$, it is sufficient to prove that for all k large enough,

$$p^*\left(f_{\phi(k)}(a)\right) \geqslant p^*\left(g_{\psi(k)}(a)\right).$$

Following Assumption C_{or}, there exist $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(z^*(a))$. Let $k \geqslant k_0$, we let $i := \phi(k)$ and $j := \psi(k)$. Since $z^*(a) \in w\text{-ls}\, H^n(a)$, for each $n \in \mathbb{N}$, there exists $z^n \in H^n(a)$ such that $z^*(a) \in w\text{-ls}\,\{z^n\}$.

We assert that there exists $\alpha \in \mathcal{I}$, such that for all $n \in \mathbb{N}$,

$$(f_i(a), g_j(a)) \in P_a^{\alpha(n)}(z^{\alpha(n)}) \times Y^{\alpha(n)}(a).$$

Indeed, for all $n \geqslant k$, $(f_i(a), g_j(a)) \in X^n(a) \times Y^n(a)$. Suppose that for all $\alpha \in \mathcal{I}$, there exist $\beta \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N} \quad z^{\alpha \circ \beta(n)} \in R_i(a).$$

Applying Assumption C_{or}, it follows that $w\text{-ls}\,\{z^n\} \subset R_i(a)$ and then $z^*(a) \in R_i(a)$, that is, $f_i(a) \notin P_a(z^*(a))$. Contradiction. It follows that there exists $\alpha \in \mathcal{I}$, such that for all $n \in \mathbb{N}$, $(f_i(a), g_j(a)) \in P_a^{\alpha(n)}(z^{\alpha(n)}) \times Y^{\alpha(n)}(a)$.

Thus applying (i) of Fact 3.5.3, we obtain that, for all n large enough,

$$p^{\alpha(n)}\left(f_i(a) - g_j(a)\right) \geqslant 0.$$

Since $(p^n)_{n \in \mathbb{N}}$ w^*-converges to p^*, it follows that $p^*\left(f_i(a)\right) \geqslant p^*\left(g_j(a)\right)$. □

We proved in Lemma 3.5.2 that H is graph measurable. With Assumption A we get that

$$\int_A x^*(a)d\mu(a) \in \int_{A^{pa}} \overline{\mathrm{co}}\, H(a)d\mu(a) + \int_{A^{na}} H(a)d\mu(a) = \int_A H(a)d\mu(a).$$

It follows that there exists an integrable selection $\bar{x}$ of H such that $\int_A \bar{x} = \int_A x^*$, that is $(\bar{x}, y^*, p^*)$ is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. □

The proof of Lemma 3.5.1 is a direct consequence of Claim 3.5.3 and Claim 3.5.4. □

3.6 Appendix : Mathematical auxiliary results

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a complete separable metric space.

3.6.1 Measurability of correspondences

A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for each open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a, x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$.

Following in Castaing and Valadier [5] and Himmelberg [17], we recall the two following classical characterizations of measurable correspondences.

Proposition 3.6.1. *Consider $F : A \twoheadrightarrow D$ a correspondence with non-empty closed values. The following properties are equivalent.*

(i) *The correspondence F is measurable.*

(ii) *There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable selections of F such that for all $a \in A$, $F(a) = \mathrm{cl}\,\{f_n(a) \mid n \in \mathbb{N}\}$.*

(iii) *For each $x \in D$, the function $\delta_F(., x) : a \mapsto d(x, F(a))$ is measurable.*

Proposition 3.6.2. *Consider $F : A \twoheadrightarrow D$ a correspondence.*

(i) *If F has non-empty closed values then the measurability of F implies the graph measurability of F.*

(ii) *If $(A, \mathcal{A}, \mu)$ is complete then the graph measurability of F implies the measurability of F.*

(iii) *If F has non-empty closed values and $(A, \mathcal{A}, \mu)$ is complete then measurability and graph measurability of F are equivalent.*

Following Aumann [3], graph measurable correspondences (possibly without closed values) have measurable selections.

Proposition 3.6.3. *Consider F a graph measurable correspondence from A into D with non-empty values. If $(A, \mathcal{A}, \mu)$ is complete then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of measurable selections of F, such that for all $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ is dense in $F(a)$.*

3.6.2 Measurability of preference relations

Let P be a correspondence from A into $D \times D$. For each function $x : A \to D$ the *upper section relative to* x is noted $P_x : A \twoheadrightarrow D$ and is defined by $a \mapsto \{y \in D \mid (x(a), y) \in P(a)\}$. For each function $y : A \to D$ the *lower section relative to* y is noted $P^y : A \twoheadrightarrow D$ and is defined by $a \mapsto \{x \in D \mid (x, y(a)) \in P(a)\}$.

Let $X : A \twoheadrightarrow D$ be a correspondence. A *correspondence of preference relations in* X is a correspondence P from A into $D \times D$ satisfying for all $a \in A$, $P(a) \subset X(a) \times X(a)$. For each $a \in A$, we note P_a the correspondence [18] from $X(a)$ into $X(a)$ defined by $x \mapsto \{y \in X(a) \mid (x, y) \in P(a)\}$. For

[18] Remark that the graph of P_a and $P(a)$ coincide.

each $y \in X(a)$ the lower inverse image of y by P_a is noted $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$. The correspondence of preference relations P in X is graph measurable if

$$\{(a, x, y) \in A \times D \times D \mid (x, y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D) \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *Aumann measurable* if

$$\forall (x, y) \in S(X) \times S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

The correspondence of preference relations P in X is *lower graph measurable* if for all measurable selection y of X, the correspondence P^y is graph measurable, that is

$$\forall y \in S(X) \quad G_{P^y} = \{(a, x) \in A \times D \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *upper graph measurable* if for all measurable selection x of X, the correspondence P_x is graph measurable, that is

$$\forall x \in S(X) \quad G_{P_x} = \{(a, y) \in A \times D \mid (x(a), y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

We propose to compare these three concepts of measurability of preference relations.

Proposition 3.6.4. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Then graph measurability of P implies lower and upper graph measurability of P, and lower or upper graph measurability of P implies the Aumann measurability of P.*

Proof. This is a direct consequence of Projection Theorem in Castaing and Valadier [5]. □

Under additional assumptions, the converse is true.

Proposition 3.6.5. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Moreover, we suppose that for a.e. $a \in A$, $X(a)$ is a closed connected subset of D, $P(a)$ is an ordered binary relation on $X(a)$ and for each $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$. Then Aumann measurability of P implies lower and upper graph measurability of P, and the lower and upper graph measurability of P implies the graph measurability of P.*

The proof of Proposition 3.6.5 is given in Martins Da Rocha [19]. A direct corollary of Proposition 3.6.2 is the following result.

Proposition 3.6.6. *If for all $a \in A$, for all $y \in X(a)$, $P^{-1}(a, y)$ is d-open in $X(a)$, then P is lower graph measurable if and only if for all measurable selection $x \in S(X)$ the correspondence R_x is measurable.*

3.6.3 Integration of correspondences

In this subsection, $(A, \mathcal{A}, \mu)$ is supposed to be finite and complete. If $F : A \twoheadrightarrow \mathbb{L}$ is a correspondence from A to $\mathbb{L}$, the set of integrable selections of F is noted $S^1(F)$. We note F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a) d\mu(a) := \left\{ v \in \mathbb{L} \;\middle|\; \exists x \in S^1(F) \quad v = \int_A x(a) d\mu(a) \right\}$.

Proposition 3.6.7. *Consider $F : A \twoheadrightarrow \mathbb{L}$ a graph measurable correspondence. If F_Σ is non-empty, we let $G : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined by*

$$\forall a \in A \quad G(a) := s - \mathrm{cl}\left[\overline{\mathrm{co}}\, F(a) + A(F_\Sigma)\right].$$

If F_Σ is non-empty and closed convex then $G_\Sigma = F_\Sigma$, and for all $p \in \mathbb{L}'$, if there exists an integrable selection g^* of G such that for a.e. $a \in A$, $p(g^*(a)) = \sup p(G(a))$, then there exists an integrable selection f^* of F satisfying for a.e. $a \in A$, $p(f^*(a)) = \sup p(F(a))$ and $\int_A f^* = \int_A g^*$.

Proof. Since $(A, \mathcal{A}, \mu)$ is complete, following Proposition 3.6.2, the correspondence F is measurable. Following Theorem 8.2.2 in Aubin and Frankowska [2], the correspondence G is measurable with s-closed-values. Once again applying Proposition 3.6.2, G is graph measurable and $F_\Sigma \subset G_\Sigma$. Moreover if $p \in \mathbb{L}'$ then

$$\forall a \in A \quad \sup p\,(G(a)) = \sup p\,(F(a)) + \sup p\,(A(F_\Sigma))\,.$$

Note that, since $A(F_\Sigma)$ is a cone containing zero, $\sup p\,(A(F_\Sigma)) \in \{0, \infty\}$.

Suppose now that F_Σ is closed convex and that there exists $v \in G_\Sigma$ such that $v \notin F_\Sigma$. Since F_Σ is closed convex, by a separation argument there exists $p \in \mathbb{L}'$ with $p \neq 0$ such that $p\,(v) > \sup p\,(F_\Sigma)$. It follows that $\sup p\,(A(F_\Sigma)) = 0$ and following Proposition [19] 6 in Hildenbrand [15],

$$\sup p\,(F_\Sigma) = \int_A \sup p\,(F(a))\, d\mu(a) = \int_A \sup p\,(G(a)) = \sup p\,(G_\Sigma)\,.$$

Thus $p\,(v) > \sup p\,(G_\Sigma)$ and this contradicts the fact that $v \in G_\Sigma$. The second part of Proposition 3.6.7 is a direct consequence of the previous result. □

Theorem 3.6.1. *Suppose F is an integrably bounded correspondence, with non-empty, w-compact and convex values. If $(f^n)_{n\in\mathbb{N}}$ is a sequence of integrable selections of F, then there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and $f^* \in S^1(F)$ an integrable selection of F, such that*

$$\int_A f^*(a)d\mu(a) = \lim_{n\to\infty} \int_A f^{\phi(n)}(a)d\mu(a),$$

and

$$\begin{aligned} &\text{for a.e. } a \in A^{na} \quad f^*(a) \in \overline{\text{co}}\, w - ls\{f^{\phi(n)}(a)\} \\ &\text{for all } a \in A^{pa} \quad f^*(a) \in w - ls\{f^{\phi(n)}(a)\}, \end{aligned}$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$.

Proof. For each $n \in \mathbb{N}$, we let $v^n := \int_A f^n$. Following Corollary 2.6 in Diestel, Ruess and Schachermayer [10] and Theorem 15, p. 422 in Dunford and Schwartz [12], the sequence $(v^n)_{n\in\mathbb{N}}$ is relatively compact. Applying Lemma 6.6 in Podczeck [20] or Corollary 4.4 in Balder and Hess [4], we get the desired result. □

For more precisions about measurability and integration of correspondences, we refer to papers [25] and [26] of Yannelis.

3.6.4 Separation of $\mathbb{Q}$-convex sets

Let $(\mathbb{L}, \tau)$ be a topological vector space. A set G is called $\mathbb{Q}$-convex if for all $x, y \in G$, for all $t \in [0,1] \cap \mathbb{Q}$, $tx + (1-t)y \in G$. The $\mathbb{Q}$-convex hull of a set G is the smallest $\mathbb{Q}$-convex set containing G. We present hereafter a result of decentralization for a $\mathbb{Q}$-convex set.

Proposition 3.6.8. *Let $(\mathbb{L}, \tau)$ be a topological vector space and G be a $\mathbb{Q}$-convex subset with a τ-interior point and such that $0 \notin G$. Then there exists a non-zero continuous linear functional $p \in (L, \tau)'$ such that*

$$\forall x \in G \quad p(x) \geqslant 0.$$

Proof. The interior $\operatorname{int} G$ of G is a non-empty and $\mathbb{Q}$-convex subset of $\mathbb{L}$. Let $x \in G$, for each $\lambda \in [0,1[\cap\mathbb{Q}$, $\lambda x + (1-\lambda)u \in \operatorname{int} G$, if $u \in \operatorname{int} G$. It follows that

$$\operatorname{int} G \subset G \subset \operatorname{cl} \operatorname{int} G.$$

Since $\operatorname{int} G$ is τ-open, it is in fact convex. Now $0 \notin \operatorname{int} G$ and we can apply a convex Separation Theorem to provide the existence of a non-zero continuous linear functional $p \in (\mathbb{L}, \tau)'$ such that for all $x \in \operatorname{int} G$, $p(x) \geqslant 0$. With a limit argument, we prove that for all $x \in G$, $p(x) \geqslant 0$. □

[19] Following Podczeck [21], this latter result is stated in terms of $\mathbb{R}^n$-valued correspondences. However, as can be seen from its proof, it generalizes directly to the context of a separable Banach space.

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Chapitre 4

Existence d'équilibres avec double infinité, propreté uniforme et des préférences non ordonnées

Résumé

L'approche introduite au chapitre 2 est maintenant appliquée pour démontrer l'existence d'un équilibre de Walras pour des économies avec un espace mesuré d'agents et des biens différenciés. Notre approche, basée sur la discrétisation des correspondances (ou multifonctions) mesurables, nous permet de démontrer l'existence d'un équilibre pour des économies avec des préférences non ordonnées et un secteur productif non trivial. Notre résultat d'existence généralise les résultats (théorèmes 1.a and 3.a) d' Ostroy and Zame [31] ainsi que (dans le cadre convexe) ceux de Podczeck [33] (théorème 5.3). En particulier les hypothèse classiques sur les taux marginaux de substitutions sont remplacées par l' hypothèse plus faible de propreté uniforme.

Mots-clés : *Espace mesuré d'agents, biens différenciés, préférences non ordonnées, propreté uniforme et discrétisation des correspondances mesurables.*

Existence of equilibria for large square economies with non-ordered preferences and uniform properness

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16th June 2002

Abstract

The Approach of Martins Da Rocha [28] is applied to provide a Walrasian equilibria existence result for economies with a measure space of agents and differentiated commodities. The approach proposed in this paper, based on the discretization of measurable correspondences, allows us to provide an existence result (Theorem 4.3.1) for economies with a non-trivial production sector and with possibly non-ordered preferences. Our existence result generalizes existence results (Theorem 1.a and 3.a) in Ostroy and Zame [31], and (in the framework of convex preferences) in Podczeck [33] (Theorem 5.3). In particular existence of equilibria is guaranteed under uniform properness conditions which are weaker than usual conditions on marginal rates of substitutions.

Keywords : *Measure space of agents, differentiated commodities, non-ordered preferences, uniform properness and discretization of measurable correspondences.*

4.1 Introduction

In the framework of differentiated commodities there exist, among others, two approaches to model economies with infinitely many agents (or consumers). In Mas-Colell [29], Jones [25] and Podczeck [34], economies are described by distributions on the space of agents' characteristics. Following Ostroy and Zame [31], Podczeck [33] and Cornet and Médecin [14], we describe an economy as a mapping from a measure space of agents to the space of agents' characteristics.

The purpose of this paper is to provide a proof of the existence of an equilibrium for economies with a measure space of agents, a finite set of producers and infinitely many differentiated commodities. The approach proposed in this paper, based on the *discretization* of measurable correspondences, allows us to provide an existence result (Theorem 4.3.1) for economies with a non-trivial production sector and with possibly non-ordered preferences. The measure space of agents is not supposed to be purely non-atomic, then we encompass the finite agents' set-up. Moreover, our approach allows for more general consumption sets than the positive cone and following the direction introduced by Podczeck in [34], the *uniform substitutability* assumptions of Mas-Colell [29], Jones [25] and Ostroy and Zame [31], are replaced by *uniform properness* assumptions. Our *uniform properness* assumptions are inspired from those presented in Podczeck [32] and in Florenzano and Marakulin [21], and they generalize uniform properness assumptions presented in Podczeck [34].

The existence proof in Mas-Colell [29], Jones [25], Ostroy and Zame [31] and Cornet and Médecin [14] consists of a limit argument based on equilibria in economies with finitely many commodities. For economies with finitely many agents, Aliprantis and Brown [2] first underline the central role of lattice structure for the commodity and price spaces (see also [3, 4, 5]). For economies with infinitely many agents, Rustichini and Yannelis in [37] and [38], are the first to focus on the lattice structure of the commodity space to prove the equivalence between the set of Core allocations and the set of Walrasian equilibria. In order to prove the non-emptyness of the set of Walrasian equilibria, Podczeck in [34] is the first focus on the lattice structure of the commodity and price spaces for economies with infinitely many agents. He succeeded to solve the equilibrium existence problem by using fixed point arguments in infinite dimensional spaces directly, rather than to proceed by finite dimensional approximations. Our approach also focuses on the lattice structure of the commodity and price spaces. In order to use the recent results establishing existence of equilibria for economies with finitely many agents (e.g. in Podczeck [32], Tourky [39], Florenzano and Marakulin [21] and many others papers [1], [9], [11], [18], [27]), our approach consists of a limit argument based on equilibria for economies with finitely many agents. If $\mathcal{E}$ is an economy with a measure space of agents, we propose to construct a sequence

$(\mathcal{E}^n)_{n\in\mathbb{N}}$ of economies with an increasing but finite set of agents, *converging* to the economy $\mathcal{E}$. We then are able to apply an equilibria existence result (provided in this paper) to each economy $\mathcal{E}^n$, in order to obtain a sequence of quasi-equilibria which will converge to a quasi-equilibrium of the initial economy $\mathcal{E}$.

The paper is organized as follows. In Section 4.2, we set the main definitions and notations. In Section 4.3 we define the model of large square economies, we introduce the concepts of equilibria, we give the list of assumptions that economies will be required to satisfy and finally, we present the existence result (Theorem 4.3.1). The Section 4.4 is devoted to the mathematical *discretization* of measurable correspondences. The proof of the main theorem (Theorem 4.3.1) is then given in Section 4.5. The equilibria existence result for economies with finitely many agents is provided in Section 4.6. The last section is devoted to mathematical auxiliary results.

4.2 Notations and definitions

Consider (E,τ) a topological vector space. If $X \subset E$ is a subset, then the τ-interior of X is noted $\tau\text{-int}\, X$, the τ closure of X is noted $\tau\text{-cl}\, X$. The convex hull of X is noted $\operatorname{co} X$ and the τ-closed convex hull of X is noted $\tau\text{-}\overline{\operatorname{co}}\, X$. If X is convex then we let $A(X) = \{v \in M(T) \mid X + \{v\} \subset X\}$ be the asymptotic cone of X and we let $A_\tau(X)$ be the set of elements $x \in L$ such that $x = \tau\text{-}\lim_n \lambda_n x_n$ where $(\lambda_n)_{n\in\mathbb{N}}$ is a real sequence decreasing to 0 and $(x_n)_{n\in\mathbb{N}}$ is a sequence in X. Note that we always have $A(X) \subset A_\tau(X)$, and if X is τ-closed convex, then $A(X) = A_\tau(X)$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of subsets of E, the τ *sequential upper limit* of $(C_n)_{n\in\mathbb{N}}$, is denoted $\tau\text{-ls}\, C_n$ and is defined by

$$\tau\text{-ls}\, C_n := \{x \in E \mid x = \tau\text{-}\lim x_k\,, \quad x_k \in C_{n(k)}\}.$$

Let T be any compact metric space. The set of all continuous functions on T is noted $C(T)$ and the set of all finite signed Borel measures on T is noted $M(T)$. Note that $C(T)$ and $M(T)$, endowed with their natural positive cones $C(T)_+$ and $M(T)_+$, are vector lattices. Given elements x, y of $C(T)$ or of $M(T)$, x^+, x^-, $|x|$, $x \vee y$, and $x \wedge y$ have the usual lattice theoretical meaning. A subset $Z \subset M(T)$ is a lattice if for all $z \in Z$, z^+ and z^- still lie in Z. If $p \in C(T)$, then $\|p\|_\infty$ denotes the sup-norm of p. If $x \in M(T)$, then $\|x\|$ denotes the variation norm of x, that is $\|x\| = |x|(T) = x^+(T) + x^-(T)$. Following the Riesz representation theorem, $M(T)$ is the topological dual of $(C(T), \|.\|_\infty)$. The natural dual pairing $\langle C(T), M(T)\rangle$ is defined by

$$\forall (p,x) \in C(T) \times M(T) \quad \langle p, x\rangle = \int_T p(t)dx(t).$$

If $p \in C(T)$, then $p > 0$ means that ($p \in C(T)_+$ and $p \neq 0$), and $p \gg 0$ means that for each $t \in T$, $p(t) > 0$. If $x \in M(T)$, then $x \gg 0$ means that for all $p \in C(T)$, if $p > 0$ then $\langle p, x\rangle > 0$. By the support of $x \in M(T)$, denoted $\operatorname{supp} x$, we mean the smallest closed subset F of T such that $|x|(T \setminus F) = 0$. Note that $x \in M(T)$ satisfies $x \gg 0$ if and only if ($x > 0$ and $\operatorname{supp} x = T$). Given any $t \in T$, we write δ_t for the Dirac measure at t, and we note 1_K the unit constant function on T, i.e. $1_K(t) = 1$ for each $t \in T$.

The weak topology $\sigma(M(T), C(T))$ on $M(T)$ is noted w^*, the Mackey topology $\tau(M(T), C(T))$ is noted τ^* and we note bw^* the strongest topology on $M(T)$ agreeing with the w^* topology on every w^*-compact set. The Borel σ-algebra of $(M(T), w^*)$ and of $(M(T), bw^*)$ coincide and is noted $\mathcal{B}$.

We consider $(A, \mathcal{A}, \mu)$ a finite measure space, that is, A is a set, $\mathcal{A}$ is a σ-algebra of subsets of A and μ is a finite measure on $\mathcal{A}$. The measure space $(A, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all μ-negligible [1] subsets of A. A function f from A to $M(T)$ is *measurable* if for all $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. Note that f is measurable if and only if it is *Gelfand measurable*, that is, for each $p \in C(T)$, the real valued function $\langle p, f(.)\rangle$ is measurable. A measurable function f from A to $M(T)$ is *Gelfand integrable* if for each $p \in C(T)$, the real valued function $\langle p, f(.)\rangle$ is integrable. Then there exists a unique element $x \in M(T)$, satisfying for each $p \in C(T)$, $\langle p, x\rangle = \int_A \langle p, f(a)\rangle\, d\mu(a)$. The element x is noted

[1] A set N is μ-negligible if there exists $E \in A$ such that $N \subset E$ and $\mu(E) = 0$.

$\int_A f(a)d\mu(a)$. A measurable function f from A to $M(T)$ is *norm integrable* if $\|f(.)\| : a \mapsto \|f(a)\|$ is integrable. Note that norm integrability implies Gelfand integrability and if f has its values in $M(T)_+$ then the converse is true. A sequence (f_n) of measurable functions from A to $M(T)$ is *integrably bounded* if there exists an integrable function h from A to $\mathbb{R}_+$ such that for a.e. $a \in A$, for all $n \in \mathbb{N}$, $\|f_n(a)\| \leqslant h(a)$. If $F : A \twoheadrightarrow M(T)$ is a correspondence then $f : A \to M(T)$ is a measurable selection of F if f is measurable and satisfies for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$ and the set of Gelfand integrable selections of F is noted $S^1(F)$.

Let X be a space and $P \subset X \times X$ be a binary relation on X. The relation P is irreflexive if $(x,x) \notin P$, for all $x \in X$. The relation P is transitive if $[(x,y) \in P$ and $(y,z) \in P]$ implies $(x,z) \in P$, for all $(x,y,z) \in X^3$. The relation P is negatively transitive if $[(x,y) \notin P$ and $(y,z) \notin P]$ implies $(x,z) \notin P$, for all $(x,y,z) \in X^3$. The relation P is a partial order it is irreflexive and transitive. The relation P is an order if it is irreflexive, transitive and negatively transitive. When P is an order, it is usually noted $\succ$ and $X^2 \setminus P$ is noted $\preceq$. Note that when P is an order, then $\preceq$ is transitive, reflexive ($x \preceq x$ for all $x \in X$) and complete (for all $(x,y) \in X^2$ either $x \preceq y$ or $y \preceq x$).

4.3 The model, the equilibrium concepts and the assumptions

4.3.1 The Model

We consider a compact metric space T, a complete finite measure space $(A, \mathcal{A}, \mu)$ and a finite set J. Moreover, we consider, for each $j \in J$, an integrable positive function θ_j from A to $\mathbb{R}_+$, satisfying $\int_A \theta_j = 1$, and a set $Y_j \subset M(T)$, a Gelfand integrable function e from A to $M(T)$, a correspondence X from A into $M(T)$ and a correspondence of preferences P in X, that is, P is a correspondence from A into $M(T) \times M(T)$ such that for all $a \in A$, $P(a) \subset X(a) \times X(a)$ and $P(a)$ is irreflexive.

A large square economy $\mathcal{E}$ with differentiated commodities, is a list

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle C(T), M(T) \rangle, (X, P, e), (Y_j, \theta_j)_{j \in J}).$$

The commodity space of $\mathcal{E}$ is represented by $M(T)$. Each point of T has the interpretation of representing a complete description of all characteristics of a certain commodity. Let $x \in M(T)$ be a commodity bundle, then for each Borel set $B \subset T$, $x(B)$ specifies the total amount of commodities having their characteristics in B. Note that since we let every element of $M(T)$ represent a possible commodity bundle, we assume, as in the models of Jones [25, 26] and Ostroy and Zame [31] but different to those of Mas-Colell [29] and Cornet and Médecin [14], that all commodities are perfectly divisible.

The natural dual pairing $\langle C(T), M(T) \rangle$ is interpreted as the *price-commodity* pairing. If $p \in C(T)$, then for each $t \in T$, $p(t)$ is interpreted as the value (or price) of one unit of the commodity with characteristic t.

The set of agents (or consumers) is represented by A, the set $\mathcal{A}$ represents the set of admissible coalitions, and the number $\mu(E)$ represents the fraction of consumers which are in the coalition $E \in \mathcal{A}$.

For each agent $a \in A$, the consumption set is represented by $X(a) \subset M(T)$ and the preferences are represented by the binary relation $P(a) \subset X(a) \times X(a)$. We define the correspondence [2] $P_a : X(a) \twoheadrightarrow X(a)$ by $P_a(x) = \{x' \in X(a) \mid (x, x') \in P(a)\}$. In particular, if $x \in X(a)$ is a consumption bundle, $P_a(x)$ is the set of consumption bundles strictly preferred to x by the agent a. The set of consumption allocations (or plans) of the economy is the set $S^1(X)$ of Gelfand integrable selections of X. The aggregate consumption set X_Σ is defined by

$$X_\Sigma := \int_A X(a)d\mu(a) := \left\{ v \in M(T) \;\middle|\; \exists x \in S^1(X) \quad v = \int_A x(a)d\mu(a) \right\}.$$

The initial endowment of the consumer $a \in A$ is represented by the commodity bundle $e(a) \in M(T)$. We note $\omega := \int_A e(a)d\mu(a)$ the aggregate initial endowment.

[2] Note that the binary relation $P(a)$ coincide with the graph of the correspondence P_a.

The production sector of the economy $\mathcal{E}$ is represented by a finite set J of firms with production sets $(Y_j)_{j\in J}$, where for every $j \in J$, $Y_j \subset M(T)$. The profit made by the firm $j \in J$ is distributed among the consumers following the share function θ_j. For each $j \in J$, $\theta_j : A \to [0,+\infty[$ satisfies $\int_A \theta_j d\mu = 1$. The set of production allocations (or plans) of the economy is the set $S^1(Y) = \prod_{j\in J} Y_j$. The aggregate production set Y_Σ is defined by $Y_\Sigma := \sum_{j\in J} Y_j$.

4.3.2 The Equilibrium Concepts

Definition 4.3.1. A *Walrasian equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $S^1(X) \times S^1(Y) \times C(T)$ such that $p^* \neq 0$ and satisfying the following properties.

(a) For almost every $a \in A$,

$$\langle p^*, x^*(a)\rangle = \langle p^*, e(a)\rangle + \sum_{j\in J}\theta_j(a)\left\langle p^*, y_j^*\right\rangle$$

and

$$x \in P_a(x^*(a)) \Longrightarrow \langle p^*, x\rangle > \langle p^*, x^*(a)\rangle .$$

(b) For every $j \in J$,

$$y \in Y_j \Longrightarrow \langle p^*, y\rangle \leqslant \left\langle p^*, y_j^*\right\rangle .$$

(c)

$$\int_A x^*(a)d\mu(a) = \int_A e(a)d\mu(a) + \sum_{j\in J} y_j^*.$$

A *Walrasian quasi-equilibrium* of an economy $\mathcal{E}$ is an element $(x^*, y^*, p^*) \in S^1(X)\times S^1(Y)\times C(T)$ such that $p^* \neq 0$ and which satisfies the conditions (b) and (c) together with

(a') for almost every $a \in A$,

$$\langle p^*, x^*(a)\rangle = \langle p^*, e(a)\rangle + \sum_{j\in J}\theta_j(a)\left\langle p^*, y_j^*\right\rangle$$

and

$$x \in P_a(x^*(a)) \Longrightarrow \langle p^*, x\rangle \geqslant \langle p^*, x^*(a)\rangle .$$

A Walrasian equilibrium of a production economy $\mathcal{E}$ is clearly a Walrasian quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, a classical condition on $\mathcal{E}$ under which a Walrasian quasi-equilibrium is in fact a Walrasian equilibrium.

Remark 4.3.1. Let (x^*, y^*, p^*) be a quasi-equilibrium of an economy $\mathcal{E}$. If for almost every agent $a \in A$, $X(a)$ is convex, the strict-preferred set $P_a(x^*(a))$ is w^*-open in $X(a)$ and

$$\inf \langle p^*, X(a)\rangle < \langle p^*, e(a)\rangle + \sum_{j\in J}\theta_j(a) \sup \langle p^*, Y_j\rangle$$

then (x^*, y^*, p^*) is a Walrasian equilibrium of $\mathcal{E}$. In particular, if $p^* \gg 0$ then the last condition is automatically valid if for almost every agent $a \in A$,

$$\left(\{e(a)\} + \sum_{j\in J}\theta_j(a)Y_j - X(a)\right) \cap M(T)_+ \neq \emptyset.$$

4.3.3 The Assumptions

We present the list of assumptions that the economy $\mathcal{E}$ will be required to satisfy.

Assumption (C). ***[Consumption Side]*** *For almost every agent $a \in A$, the consumption set $X(a)$ is w^*-closed and convex ; for each bundle $x \in X(a)$, $P_a(x)$ is τ^*-open in $X(a)$, $P_a^{-1}(x)$* [3] *is w^*-open in $X(a)$, $x \notin \operatorname{co} P_a(x)$, and if a belongs to the non-atomic* [4] *part of $(A, \mathcal{A}, \mu)$, then $X(a) \setminus P_a^{-1}(x)$ is convex.*

Remark 4.3.2. Note that when $P(a)$ is ordered, then following the notations of Section 4.2, $X(a) \setminus P_a^{-1}(x) = \{y \in X(a) \mid y \succeq_a x\}$ and assuming that for all $x \in X(a)$, $\{y \in X(a) \mid y \succeq_a x\}$ is convex implies that for $x \in X(a)$, $x \notin \operatorname{co} P_a(x)$. It follows that Assumption C is implied by Assumptions E1-3 and S1 in Podczeck [33] and by Assumptions P1-4 for economically thick markets of Ostroy and Zame [31].

Assumption (M). ***[Measurability]*** *The correspondence X is graph measurable, that is,*

$$\{(a, x) \in A \times M(T) \mid x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}$$

and the correspondence of preferences P is lower graph measurable, that is,

$$\forall y \in S(X) \quad \{(a, x) \in A \times M(T) \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}.$$

Remark 4.3.3. Under Assumption C, the correspondence X is closed valued and for all $x \in S(X)$, the correspondence [5] R_x is w^*-closed valued. Under the following Assumption B and Propositions 4.7.3 and 4.7.7, Assumption M is valid if and only if the correspondence X is measurable [6] and for all measurable selection $x \in S(X)$, the correspondence R_x is measurable. It follows that if A is a finite set and $\mathcal{A} = 2^A$, Assumption M is automatically valid.

Remark 4.3.4. In Podczeck [33], the correspondences X and P are supposed to be graph measurable. Following Proposition 4.7.5, Assumption M is then valid. In Ostroy and Zame [31], it is assumed that preferences are Aumann measurable, applying Proposition 4.7.6, Assumption M is then valid.

Assumption (P). ***[Production side]*** *The aggregate production set Y_Σ is a bw^*-closed and convex subset of $M(T)$.*

Remark 4.3.5. For economies with finitely many commodities, Hildenbrand [22] already used Assumption P. For economies with finitely many consumers, Jones [26] supposed that Y_Σ is w^*-closed and convex, this is equivalent to Assumption P since the topology bw^* is locally convex and compatible with the duality $\langle M(T), C(T)\rangle$.

Assumption (S). ***[Survival]*** *For almost every $a \in A$,*

$$0 \in \left(\{e(a)\} + \sum_{j \in J} \theta_j(a) \overline{\operatorname{co}}\, Y_j + A(Y_\Sigma) - X(a) \right).$$

Remark 4.3.6. Assumption S means that we have compatibility between individual needs and resources. In the literature of economies with differentiated commodities, this assumption is automatically valid since initial endowments are supposed to lie in the consumption set and since inaction is supposed to be a possible production plan.

Assumption (MON). ***[Monotonicity]*** *For almost every agent $a \in A$, the preference relation $P(a)$ is monotone, that is*

$$\forall m \in M(T)_+ \quad \exists \alpha > 0 \quad x + \alpha m \in P_a(x) \cup \{x\}.$$

[3] We let $P_a^{-1}(x) = \{y \in X(a) \mid y \in P_a(x)\ \}$.

[4] An element $E \in \mathcal{A}$ is an atom of $(A, \mathcal{A}, \mu)$ if $\mu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\mu(B) = 0$ or $\mu(E \setminus B) = 0$.

[5] Following Section 4.7, $R_x : A \to M(T)$ is defined by $a \mapsto \{y \in X(a) \mid (y, x) \notin P(a)\}$.

[6] A correspondence $F : A \twoheadrightarrow M(T)$ is measurable if for all w^*-open set V the set $F^-(V) = \{a \in A \mid F(a) \cap V \neq \emptyset\}$ is measurable.

Remark 4.3.7. Usually in the literature, it is supposed that for almost every agent $a \in A$, for all bundle $x \in X(a)$,

$$\{x\} + M(T)_+ \subset P_a(x) \cup \{x\}.$$

Assumption (E). ***[Endowments]*** *There exists $\overline{v} \in X_\Sigma$ and $\overline{u} \in Y_\Sigma$ such that $\omega + \overline{u} - \overline{v} \gg 0$.*

Remark 4.3.8. That is, there exists an aggregate production plan $\overline{u} \in Y_\Sigma$ such that together with the aggregate initial endowment, all commodities are available in the aggregate consumption set. Usually in the literature of differentiated commodities, the consumption sets are suppose to coincide with the positive cone. It follows that if it is assumed that $\omega \gg 0$ and $0 \in Y_\Sigma$ (e.g. in [25, 26, 31, 33]) or that $\omega + \overline{u} \gg 0$ (in [34]), then Assumption E is valid.

Assumption (B). ***[Bounded]*** *The correspondence X of consumption sets is norm integrably bounded from below [7], the initial endowment function e is norm integrable and the aggregate set of free production $Y_\Sigma \cap M(T)_+$ is norm bounded.*

Remark 4.3.9. Following Assumption B there exists a norm integrable function $\underline{x} : A \to M(T)$ such that for a.e. $a \in A$, $X(a) \subset \{\underline{x}(a)\} + M(T)_+$. Usually in the literature the consumptions sets are supposed to coincide with the positive cone $M(T)_+$ and initial endowments are suppose to be Gelfand integrable and to lie in the positive cone. Note that if $\underline{x}$ is a Gelfand integrable function from A to $-M(T)_+$ and e is a Gelfand integrable function such that for all $a \in A$, $e(a) \geqslant \underline{x}(a)$ then $\underline{x}$ and e are norm integrable. Hildenbrand in [22] and Podczeck in [34] assumed that there is no free production, that is $Y_\Sigma \cap M(T)_+ = \{0\}$.

Assumption (WSS). *For almost every agent $a \in A$,*

$$\left(\{e(a)\} + \sum_{j \in J} \theta_j(a) \overline{\mathrm{co}}\, Y_j + A(Y_\Sigma) - X(a) \right) \cap M(T)_+ \neq \{0\}.$$

Remark 4.3.10. Under Assumption C and WSS, each quasi-equilibrium (x^*, y^*, p^*) with $p^* \gg 0$ is in fact a Walrasian equilibrium. This assumption may be replaced by standard irreducibility conditions adapted to our context, see Podczeck [35].

Assumption (UP). ***[Uniform Properness]*** *There exists a bw*-open cone Γ, such that $\Gamma \cap M(T)_+ \neq \emptyset$ and such that for almost every $a \in A$, for every $j \in J$, for every $(x, y) \in X(a) \times Y_j$,*

(a) *there exists a subset A^a_x of $M(T)$, radial [8] at x, such that*

$$(\{x\} + \Gamma) \cap \{z \in M(T) \mid z \geqslant x \wedge e(a)\} \cap A^a_x \subset \overline{\mathrm{co}}\, P_a(x) \ ;$$

(b) *there exists a subset A^j_y of $M(T)$, radial at y, such that*

$$(\{y\} - \Gamma) \cap \{z \in M(T) \mid z \leqslant y \vee 0\} \cap A^j_y \subset \overline{\mathrm{co}}\, Y_j.$$

Remark 4.3.11. This assumption is borrowed from the F-properness assumption introduced by Podczeck [32] for pure exchange economies with finitely many agents and adapted to production economies by Florenzano and Marakulin [21]. For refinements about the properness conditions used in the literature, we refer to Aliprantis, Tourky and Yannelis [7].

Remark 4.3.12. In Assumption UP, property (a) is close to the asymmetric part of the uniform properness for exchange economies developed in Mas-Colell [30] and property (b) is close to the asymmetric part of the uniform properness developed for production economies in Richard [36].

[7] That is there exists a norm integrable function $\underline{x} : A \to M(T)$ such that for a.e. $a \in A$, $X(a) \subset \{\underline{x}(a)\} + M(T)_+$.

[8] A subset $R \subset M(T)$ is radial at $x \in R$ if for all $v \in M(T)$, there exists $\lambda > 0$ such that the segment $[x, x + \lambda v]$ still lie in R.

Remark 4.3.13. Assumption UP is weaker than Assumptions C3 and P4 in Podczeck [34], since the radial sets A_x^a and A_y^j are supposed to coincide with $M(T)$. Hence following Propositions 3.2.1 and 3.3.1 in [34], Assumption UP is weaker than usual assumptions about marginal rates of substitution in models of commodity differentiation, e.g. in Jones [25, 26], Ostroy and Zame [31] and Podczeck [33].

Remark 4.3.14. Following the proof of the existence theorem, we can replace the condition (b) by the following condition (b').

(b') For all $u \in Y_\Sigma$, there exists a subset A'_u of $M(T)$, radial at u, such that

$$(\{u\} - \Gamma) \cap \{z \in M(T) \mid z \leqslant u \vee 0\} \cap A'_u \subset Y_\Sigma.$$

4.3.4 Existence Result

Theorem 4.3.1. *If $\mathcal{E}$ is an economy satisfying Assumptions **C, M, P, S, B, MON, E** and **UP**, then there exists a quasi-equilibrium (x^*, y^*, p^*), with $p^* \gg 0$. If moreover $\mathcal{E}$ satisfies **WSS**, then (x^*, y^*, p^*) is a Walrasian equilibrium.*

Remark 4.3.15. This existence result extends to economies with a non-trivial production sector and with possibly non-ordered preferences, existence results (Theorem 1.a and 3.a) in Ostroy and Zame [31], and (in the framework of convex preferences) in Podczeck [33] (Theorem 5.3). Theorem 4.3.1 allows for more general consumption sets than the positive cone and the Uniform Properness Assumption is weaker than usual assumptions about marginal rates of substitution in models of commodity differentiation, e.g. in Jones [25, 26], Ostroy and Zame [31] and Podczeck [33].

Remark 4.3.16. In Tourky and Yannelis [40], it is proved (since $(M(T), \|.\|)$ is not separable) that we can construct an economy (with a measure space of agents) satisfying all the usual assumptions but for which no Bochner integrable Walrasian equilibrium exists. Note however that in this model, allocations are only required to be Gelfand integrable.

Remark 4.3.17. As it is frequently done in the literature, instead of Assumption E, we can assume that the aggregate endowment is a uniform properness vector of the economy, or more generally:

Assumption (E'). *There exists $\overline{v} \in X_\Sigma$ and $\overline{u} \in Y_\Sigma$ such that $\omega + \overline{u} - \overline{v} \in \Gamma \cap M(T)_+$.*

4.4 Discretization of measurable correspondences

4.4.1 Notations and definitions

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a separable metric space. A function $f : A \to D$ is *measurable* if for all open set $G \subset D$, $f^{-1}(G) \in \mathcal{A}$ where $f^{-1}(G) := \{a \in A \mid f(a) \in G\}$. A correspondence $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) \in \mathcal{A}$ where $F^-(G) := \{a \in A \mid F(a) \cap G \neq \emptyset\}$.

Definition 4.4.1. A partition $\sigma = (A_i)_{i \in I}$ of A is a *measurable partition* if for all $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of A is *subordinated to the partition* σ if there exists a family $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ such that $A^\sigma = \{a_i \mid i \in I\}$.

4.4.1.1 Simple functions subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i \in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the application which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i \in I} f(a_i) \chi_{A_i} ,$$

where χ_{A_i} is the characteristic [9] function associated to A_i.

[9]That is, for all $a \in A$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

Definition 4.4.2. A function $s : A \to D$ is called *a simple function subordinated* to f if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $s = \phi(\sigma, A^\sigma)(f)$.

4.4.1.2 Simple correspondences subordinated to a measurable partition

Given a couple (σ, A^σ) where $\sigma = (A_i)_{i\in I}$ is a measurable partition of A, and $A^\sigma = \{a_i \mid i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the application which maps each measurable correspondence $F : A \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i\in I} F(a_i)\chi_{A_i}.$$

Note that the sum is well defined since there exists at most one non zero factor.

Definition 4.4.3. A correspondence $S : A \to D$ is called a *simple correspondence subordinated* to a correspondence F if there exists a couple (σ, A^σ) where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $S = \psi(\sigma, A^\sigma)(F)$.

Remark 4.4.1. If f is a function from A to D, let $\{f\}$ be the correspondence from A into D, defined for all $a \in A$ by $\{f\}(a) := \{f(a)\}$. We check that

$$\psi(\sigma, A^\sigma)(F) = \{\phi(\sigma, A^\sigma)(f)\}.$$

4.4.1.3 Hyperspace

The space of all non-empty subsets of D is noted $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wisjman topology on $\mathcal{P}^*(D)$, that is the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x,.))_{x\in D}$. The Hausdorff semi-metric H_d on $\mathcal{P}^*(D)$ is defined by

$$\forall (A,B) \in \mathcal{P}^*(D) \quad H_d(A,B) := \sup\{|d(x,A) - d(x,B)| \mid x \in D\}.$$

A subset C of D is the Hausdorff limit of a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of D, if

$$\lim_{n\to\infty} H_d(C_n, C) = 0.$$

4.4.2 Approximation of measurable correspondences

Hereafter we assert that for a countable set of measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence. The proof of the following theorem is given in Martins Da Rocha [28].

Theorem 4.4.1. *Let $\mathcal{F}$ be a countable set of measurable correspondences with non-empty values from A into D and let $\mathcal{G}$ be a finite set of integrable functions from A to $\mathbb{R}$. There exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i\in I^n}$ of A, satisfying the following properties.*

(a) *Let $(A^n)_{n\in\mathbb{N}}$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $F \in \mathcal{F}$. For all $n \in \mathbb{N}$, we define the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F. The following properties are then satisfied.*

1. *For all $a \in A$, $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_{n\in\mathbb{N}}$, i.e. ,*

$$\forall a \in A \quad \forall x \in A \quad \lim_{n\to\infty} d(x, F^n(a)) = d(x, F(a)).$$

2. *If D is d-bounded then for all $x \in D$ the real valued function $d(x, F(.))$ is the uniform limit of the sequence $(d(x, F^n(.)))_{n\in\mathbb{N}}$.*
3. *If D is d-totally bounded then F is the uniform Hausdorff limit of the sequence $(F^n)_{n\in\mathbb{N}}$.*

(b) *There exists a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each $n \in \mathbb{N}$, if we let $f^n := \phi(\sigma^n, A^n)(f)$ be the simple function subordinated to each $f \in \mathcal{G}$, then*

$$\forall f \in \mathcal{G} \quad \forall a \in A \quad |f^n(a)| \leqslant 1 + \sum_{g\in\mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n\to\infty} \int_A |f^n(a) - f(a)| d\mu(a) = 0.$$

Remark 4.4.2. The property (a1) implies in particular that, if $(x^n)_{n\in\mathbb{N}}$ is a sequence of D, d-converging to $x \in D$, then

$$\forall a \in A \quad \lim_{n\to\infty} d(x^n, F^n(a)) = d(x, F(a)).$$

It follows that if F is non-empty closed valued, then property (a1) implies that

$$\forall a \in A \quad \operatorname{ls} F^n(a) \subset F(a).$$

4.5 Proof of the existence theorem

Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S, MON, E, B and UP.

Without any loss of generality [10] we can suppose that for all $a \in A$, $\underline{x}(a) = 0$ and for all $j \in J$, $0 \in Y_j$. Moreover, without any loss of generality [11], we can suppose that Assumption S is replaced by the following stronger Assumption S'

$$\text{for a.e. } a \in A \quad 0 \in \left(\{e(a)\} + \sum_{j\in J} \theta_j(a) \overline{\operatorname{co}}\, Y_j - X(a) \right).$$

Following Podczeck [32] and Holmes [24], the w^*, τ^* and bw^* topologies coincide on $D := M(T)_+$. Moreover this topology is separable and completely metrizable. We let d be a metric on D satisfying these properties.

Following Remark 4.3.3, the correspondence X is measurable. Applying Proposition 4.7.2, there exists a sequence $(f_k)_{k\in\mathbb{N}}$ of measurable selections of X such that for all $a \in A$, $X(a) := d\text{-cl}\{f_k(a) \mid k \in \mathbb{N}\}$. We let for all $k \in \mathbb{N}$, $R_k : A \twoheadrightarrow M(T)$ be the correspondence defined by $R_{f_k}(a) = \{x \in X(a) \mid f_k(a) \notin P_a(x)\}$. Then for almost every agent $a \in A$, for all $x \in X(a)$,

$$d(x, R_k(a)) > 0 \Leftrightarrow f_k(a) \in P_a(x).$$

If f is a function from A to D, then we let $\{f(.)\}$ be the correspondence from A into D defined for all $a \in A$, by $\{f(.)\}(a) := \{f(a)\}$. Note that if f is measurable then f is Gelfand integrable if and only if $\|f(.)\| : a \mapsto \|f(a)\|$ from A to $\mathbb{R}_+$ is integrable.

Let $\mathcal{G} := \{\|e(.)\|, \theta_j \mid j \in J\}$ and $\mathcal{F} := \mathcal{G} \cup \{\{e(.)\}, \{f_k(.)\}, R_k \mid k \in K\}$. Applying Theorem 4.4.1, there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of measurable partitions $\sigma^n = (A_i^n)_{i\in S^n}$ of $(A, \mathcal{A})$, and a sequence $(A^n)_{n\in\mathbb{N}}$ of finite sets $A^n = \{a_i^n \mid i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying the following properties [12].

[10] Following Assumption S, for each $j \in J$ there exists $\hat{y}_j \in Y_j$. Consider now the economy $\tilde{\mathcal{E}}$ where for each $a \in A$, $\tilde{X}(a) = X(a) - \{\underline{x}(a)\}$, for each $j \in J$, $\tilde{Y}_j = Y_j - \{\hat{y}_j\}$ and $\tilde{e}(a) = e(a) - \underline{x}(a) + \sum_{j\in J} \theta_j(a)\hat{y}_j$.

[11] Indeed, for each $a \in A$, let $\eta(a) := \sum_{j\in J} \theta_j(a)$ and let $B = \{a \in A \mid \eta(a) = 0\}$. Now consider the economy $\tilde{\mathcal{E}}$ which is the copy of $\mathcal{E}$ but with an extra producer ∞, defined by $Y_\infty = A(Y_\Sigma)$ and, if $\mu(B) = 0$ then $\theta_\infty(a) = 1/\mu(A)$ for all $a \in A$; if $\mu(B) \neq 0$, let for each $a \in B$, $\theta_\infty(a) = 1/(2\mu(B))$ and if $a \notin B$ let $\theta_\infty(a) = \eta(a)/(2\text{Card}J)$. Following Remark 4.3.14, it is straightforward to verify that the economy $\tilde{\mathcal{E}}$ satisfies Assumptions C, M, P, S', MON, E, B and UP. It is now classical to construct a quasi-equilibrium of $\mathcal{E}$ from a quasi-equilibrium of $\tilde{\mathcal{E}}$.

[12] Following notations of Section 4.4, if f is function from A to D, then for each $n \in \mathbb{N}$, $\{f(.)\}^n = \{f^n(.)\}$.

Fact 4.5.1. For all $a \in A$,

(i) for every $j \in J$ and for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} e^n(a) = e(a)\,, \quad \lim_{n\to\infty} \theta_j^n(a) = \theta_j(a) \quad \text{and} \quad \lim_{n\to\infty} f_k^n(a) = f_k(a)\ ;$$

(ii) for all sequence $(x^n)_{n\in\mathbb{N}}$ of D, d-converging to $x \in D$, for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x^n, X^n(a)) = d(x, X(a)) \quad \text{and} \quad \lim_{n\to\infty} d(x^n, R_k^n(a)) = d(x, R_k(a)).$$

(iii) if we pose $g(a) := \sum_{j\in J}\theta_j(a) + \|e(a)\|$ then g is an integrable function satisfying

$$\forall n \in \mathbb{N} \quad \max\{\theta_j^n(a), \|e^n(a)\| \mid j \in J\} \leqslant 1 + g(a)$$

and if we pose for each $n \in \mathbb{N}$, $\omega^n := \int_A e^n$ and $\vartheta_j^n := \int_A \theta_j^n$, then

$$\lim_{n\to\infty} \omega^n = \omega \quad \text{and} \quad \forall j \in J \quad \lim_{n\to\infty} \vartheta_j^n = 1.$$

4.5.1 Approximating sequence of economies

We propose to construct a sequence $(\mathcal{E}^n)_{n\in\mathbb{N}}$ of economies with finitely many consumers and differentiated commodities, *converging* to $\mathcal{E}$.

For each $n \in \mathbb{N}$, we let $\vartheta^n := \max\{\vartheta_j^n \mid j \in J\}$. Applying Fact 4.5.1, $\lim_{n\to\infty} \vartheta^n = 1$, thus, without any loss of generality, we can suppose that, for all $n \in \mathbb{N}$, $1/2 \leqslant \vartheta^n \leqslant 2$.

For each $n \in \mathbb{N}$, we note $\mathcal{E}^n$ the following economy with finitely many consumers and differentiated commodities:

$$\mathcal{E}^n = \Big(\langle C(T), M(T)\rangle\,, (X_i^n, P_i^n, e_i^n)_{i\in I^n\cup\{\infty\}}\,, \left(Y_j^n, \theta_j^n\right)_{j\in J}\Big)\,,$$

where $I^n := \{i \in S^n \mid \mu(A_i^n) \neq 0\}$. For all $j \in J$, the production set is defined by $Y_j^n := \vartheta^n Y_j$ and the shares are defined by

$$\forall i \in I^n \quad \theta_{ij}^n := \frac{1}{\vartheta^n}\mu\left(A_i^n\right)\theta_j\left(a_i^n\right) \quad \text{and} \quad \theta_{\infty j}^n := \frac{\vartheta^n - \vartheta_j^n}{\vartheta^n}.$$

The characteristics of the consumer $i \in I^n$ are defined by $X_i^n = \mu\left(A_i^n\right) X\left(a_i^n\right)$, $e_i^n = \mu\left(A_i^n\right) e\left(a_i^n\right)$ and $P_i^n = \mu\left(A_i^n\right) P\left(a_i^n\right)$. The characteristics of the consumer ∞ are defined by $X_\infty^n := D$, $e_\infty^n := 0$ and $P_\infty^n := \{(x,y) \in D^2 \mid y - x \in \Gamma\}$.

Claim 4.5.1. For all $n \in \mathbb{N}$, the economy $\mathcal{E}^n$ satisfies the assumptions of Theorem 4.6.1.

Proof. Indeed, the only assumption whose verification is not trivial is the boundedness of the set $\mathcal{A}_X(\mathcal{E}^n)$ of realizable consumption allocations. We recall that:

$$\mathcal{A}_X(\mathcal{E}^n) = \left\{ x \in \prod_{i\in I^n\cup\{\infty\}} X_i^n \;\middle|\; \sum_{i\in I^n} x_i + x_\infty - \omega^n \in \vartheta^n Y_\Sigma \right\}.$$

And it follows that

$$x \in \mathcal{A}_X(\mathcal{E}^n) \Longrightarrow \sum_{i\in I^n} x_i + x_\infty \in D \cap Z\,, \quad \text{where} \quad Z := \bigcup_{n\in\mathbb{N}} \left(\{\omega^n\} + \vartheta^n Y_\Sigma\right).$$

Since $0 \in Y_\Sigma$ and Y_Σ is convex, $\bigcup_{n\in\mathbb{N}} \vartheta^n Y_\Sigma \subset 2Y_\Sigma$ and $A_{w^*}(Z) \subset A_{w^*}(Y_\Sigma)$. Then following Assumption B, $A_{w^*}(Z) \cap A_{w^*}(D) \subset Y_\Sigma \cap M(T)_+ = \{0\}$. Applying Proposition 4.7.1, we get that, for all $x \in \mathcal{A}_X(\mathcal{E}^n)$, $\sum_{i\in I^n} x_i$ lie in a bounded set. For each $i \in I^n$, $x_i \geqslant 0$ and $\left\|\sum_{i\in I^n} x_i\right\| = \sum_{i\in I^n} \|x_i\|$. Hence $\mathcal{A}_X(\mathcal{E}^n)$ is bounded. □

Let $v \in \Gamma \cap M(T)_+$ be a properness vector and let V be a bw^*-open convex and symmetric subset of $M(T)$ such that $\{v\} + V \subset \Gamma$. Applying Claim 4.5.1, there exists a quasi-equilibrium

$$\left((x_i^n)_{i \in I^n \cup \{\infty\}}, \left(z_j^n\right)_{j \in J}, p^n \right) \in \prod_{i \in I^n \cup \{\infty\}} X_i^n \times \prod_{j \in J} Y_j^n \times C(T)$$

for the economy $\mathcal{E}^n$, with $\langle p^n, v \rangle = 1$ and $|\langle p^n, V \rangle| \leqslant 1$. Following Proposition 4.7.10, there exists a set K compact in $(C(T), \|.\|_\infty)$ such that, for all $n \in \mathbb{N}$, $p^n \in K$. For every $j \in J$, let $y_j^n := \frac{1}{\vartheta^n} z_j^n \in Y_j$. Let us then define $x^n : A \to D$, by:

$$x^n := \sum_{i \in I^n} \frac{1}{\mu(A_i^n)} x_i^n \chi_{A_i^n}.$$

We have defined a Gelfand integrable function $x^n : A \to D$ such that:

$$\forall a \in \bigcup_{i \in I^n} A_i^n \quad \langle p^n, x^n(a) \rangle = \langle p^n, e(a) \rangle + \sum_{j \in J} \theta_j^n(a) \vartheta^n \left\langle p^n, y_j^n \right\rangle \tag{4.1}$$

$$\langle p^n, x_\infty^n \rangle = \sum_{j \in J} (\vartheta^n - \vartheta_j^n) \left\langle p^n, y_j^n \right\rangle \tag{4.2}$$

$$\forall a \in \bigcup_{i \in I^n} A_i^n \quad \langle p^n, P_a^n(x^n(a)) \rangle \geqslant \langle p^n, x^n(a) \rangle \tag{4.3}$$

$$\langle p^n, P_\infty^n(x_\infty^n) \rangle \geqslant \langle p^n, x_\infty^n \rangle \tag{4.4}$$

$$\forall j \in J \quad \left\langle p^n, y_j^n \right\rangle \geqslant \langle p^n, Y_j \rangle \tag{4.5}$$

$$\int_A x^n(a) d\mu(a) + x_\infty^n = \omega^n + \vartheta^n \sum_{j \in J} y_j^n. \tag{4.6}$$

We let A_0 be the following measurable set $A_0 := \bigcup_{n \in \mathbb{N}} A \setminus (\cup_{i \in I^n} A_i^n)$. Note that $\mu(A_0) = 0$.

4.5.2 Convergence of $(x^n, y^n, p^n)_{n \in \mathbb{N}}$

Since for all $n \in \mathbb{N}$, $p^n \in K$, we can suppose (extracting a subsequence if necessary) that $(p^n)_{n \in \mathbb{N}}$ is a $\|.\|_\infty$-convergent sequence to $p^* \in K \subset C(T)$. Since, for all $n \in \mathbb{N}$, $\langle p^n, v \rangle = 1$ then $\langle p^*, v \rangle = 1$. Let us remark that following (4.3) and Assumption C, we have for all $n \in \mathbb{N}$, $p^n \geqslant 0$, and thus $p^* \geqslant 0$.

We let $G := \{-\omega^n / \vartheta^n \mid n \in \mathbb{N}\}$ and $u^n := \sum_{j \in J} y_j^n$. Following (4.6), we have, for all $n \in \mathbb{N}$,

$$u^n \in (G + M(T)_+) \cap Y_\Sigma.$$

Since G is bounded, $A_{w^*}(G + M(T)_+) = M(T)_+$. Applying Proposition 4.7.1 and Assumption B, we can conclude that the sequence $(u^n)_{n \in \mathbb{N}}$ is $\|.\|$-bounded. We can suppose (extracting a subsequence if necessary) that $(u^n)_{n \in \mathbb{N}}$ is sequence w^*-converging to $u^* \in Y_\Sigma$. It follows that there exists $y^* \in S^1(Y)$ such that $u^* = \sum_{j \in J} y_j^*$.

Claim 4.5.2. For all $j \in J$, $\lim_{n \to \infty} \left\langle p^n, y_j^n \right\rangle = \left\langle p^*, y_j^* \right\rangle$ and $\left\langle p^*, y_j^* \right\rangle = \sup \langle p^*, Y_j \rangle$.

Proof. The sequence $(p^n)_{n \in \mathbb{N}}$ is $\|.\|_\infty$-convergent to p^* and the sequence $(u^n)_{n \in \mathbb{N}}$ is w^*-convergent to u^*, it follows that

$$\lim_{n \to \infty} \langle p^n, u^n \rangle = \langle p^*, u^* \rangle.$$

Since $(\langle p^n, u^n \rangle)_{n \in \mathbb{N}}$ converges, the sequence $\left(\sum_{j \in J} \left\langle p^n, y_j^n \right\rangle \right)_{n \in \mathbb{N}}$ is bounded. For every $j \in J$, $0 \in Y_j$, hence for all $n \in \mathbb{N}$, $\left\langle p^n, y_j^n \right\rangle \geqslant 0$. It follows that, for each $j \in J$, the sequence $\left(\left\langle p^n, y_j^n \right\rangle \right)_{n \in \mathbb{N}}$ is bounded. Then passing to a subsequence if necessary, we can suppose that, for each $j \in J$, the sequence $\left(\langle p^n, y_j^n \rangle\right)_{n\in\mathbb{N}}$ converges to some $\alpha_j \geqslant 0$. We easily check that:

$$\sum_{j\in J}\alpha_j = \sum_{j\in J}\left\langle p^*, y_j^* \right\rangle .$$

Following (4.5), we have, for all $n \in \mathbb{N}$, $\langle p^n, u^n \rangle = \sup \langle p^n, Y_\Sigma \rangle$. Passing to the limit, we get that $\langle p^*, u^* \rangle = \sup \langle p^*, Y_\Sigma \rangle$. It is now routine to prove that:

$$\forall j \in J \quad \left\langle p^*, y_j^* \right\rangle = \sup \langle p^*, Y_j \rangle .$$

Moreover, since for all $n \in \mathbb{N}$, for each $j \in J$, $\left\langle p^n, y_j^n \right\rangle = \sup \langle p^n, Y_j \rangle$, we easily check that, for each $j \in J$, $\alpha_j \sup \langle p^*, Y_j \rangle$. It follows that, for each $j \in J$, $\alpha_j = \left\langle p^*, y_j^* \right\rangle$. □

Following Claim 4.5.2, the production plan $y^* \in S^1(Y)$ satisfies the condition (b) of the definition of a quasi-equilibrium for the economy $\mathcal{E}$.

Claim 4.5.3. $p^* \gg 0$.

Proof. We already proved that $p^* \geqslant 0$. Suppose that there exists $t \in T$ such that $p^*(t) = 0$. We let $B \in \mathcal{A}$ be the following set:

$$B := \left\{ a \in A \;\middle|\; \left\langle p^*, e(a) + \sum_{j\in J}\theta_j(a)\overline{y_j} - \overline{x}(a) \right\rangle > 0 \right\},$$

where $\overline{x} \in S^1(X)$ is such that $\overline{v} = \int_A \overline{x}$ and $\overline{y} \in S^1(Y)$ such that $\overline{v} = \sum_{j\in J}\overline{y}_j$. Assumption E implies that $\langle p^*, \omega + \overline{u} - \overline{v} \rangle > 0$, hence $\mu(B) > 0$.

Claim 4.5.4. For a.e. $a \in B$, $\lim_{n\to\infty} \|x^n(a)\| = +\infty$.

Proof. Let $B' \subset B$ be a measurable subset of B, with $\mu(B \setminus B') = 0$, such that all *almost everwhere* assumptions and properties are satisfied for all $a \in B'$ and such that $B' \subset A \setminus A_0$.
Let $a \in B'$. Suppose that there exists a subsequence [13] of $(x^n(a))_{n\in\mathbb{N}}$, w^*-converging to $m \in M(T)$. For every $n \in \mathbb{N}$, $x^n(a) \in X^n(a)$, it follows that, for every $n \in \mathbb{N}$, $d(x^n(a), X^n(a)) = 0$. Now applying [14] Fact 4.5.1 and using the fact that $(x^n(a))_{n\in\mathbb{N}}$ converges to m, we get that $d(m, X(a)) = 0$. Since $X(a)$ is closed, it means that $m \in X(a)$. We will now prove that:

$$\forall z \in P_a(m) \quad \langle p^*, z \rangle \geqslant \langle p^*, m \rangle .$$

Let $z \in P_a(m)$. We have that $X(a) = d\text{-cl}\{f_k(a) \mid k \in \mathbb{N}\}$, thus there exists a subsequence of $(f_k(a))_{k\in\mathbb{N}}$ [15] converging to z. But $P_a(m)$ is d-open in $X(a)$, thus there exists $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$, $f_k(a) \in P_a(m)$. To prove that $\langle p^*, z \rangle \geqslant \langle p^*, m \rangle$, it is sufficient to prove that for all k large enough, $\langle p^*, f_k(a) \rangle \geqslant \langle p^*, m \rangle$. Now, let $k \geqslant k_0$. Since $(x^n(a))_{n\in\mathbb{N}}$ is d-convergent to m, applying Fact 4.5.1,

$$\lim_{n\to\infty} d(x^n(a), R_k^n(a)) = d(m, R_k(a)).$$

Since $f_k(a) \in P_a(m)$, then $d(m, R_k(a)) > 0$ and it follows that for all n large enough, $d(x^n(a), R_k^n(a)) > 0$. Since $x^n(a) \in X^n(a)$, it follows that for all n large enough, $f_k^n(a) \in P_a^n(x^n(a))$. Applying (4.3), we obtain that, for all n large enough, $\langle p^n, f_k^n(a) \rangle \geqslant \langle p^n, x^n(a) \rangle$. Applying Fact 4.5.1, $\langle p^*, f_k(a) \rangle \geqslant \langle p^*, m \rangle$.

We will now prove that for all $z \in P_a(m)$, $\langle p^*, z \rangle > \langle p^*, m \rangle$.

Since $a \in B'$, $\left\langle p^*, e(a) + \sum_{j\in J}\theta_j(a)\overline{y_j} - \overline{x}(a) \right\rangle > 0$ and thus

$$\inf \langle p^*, X(a) \rangle \leqslant \langle p^*, \overline{x}(a) \rangle < \left\langle p^*, e(a) + \sum_{j\in J}\theta_j(a)\overline{y_j} \right\rangle \leqslant \left\langle p^*, e(a) + \sum_{j\in J}\theta_j(a)y_j^* \right\rangle .$$

[13] Still denoted $(x^n(a))_{n\in\mathbb{N}}$.
[14] We recall that in D the w^*-topology and the bw^*-topology coincide with the metric d.
[15] Still denoted $(f_k(a))_{k\in\mathbb{N}}$.

Passing to the limit in (4.1), $\inf \langle p^*, X(a)\rangle < \langle p^*, m\rangle$ and the rest of the proof is routine.

Following Assumption MON, there exists $\alpha > 0$ such that $m + \alpha\delta_t \in P_a(m)$ thus, following the previous result, we have that $\langle p^*, m + \alpha\delta_t\rangle > \langle p^*, m\rangle$, i.e. , $p^*(t) > 0$. Contradiction.

It follows that the sequence $(x^n(a))_{n\in\mathbb{N}}$ has no w^*-convergent subsequence. Hence

$$\lim_{n\to\infty} \|x^n(a)\| = +\infty.$$

□

From (4.6),

$$\int_A x^n(a)d\mu(a) + x^n_\infty = \omega^n + \vartheta^n u^n.$$

But for almost every $a \in A$, for all $n \in \mathbb{N}$, $x^n(a) \geqslant 0$, it follows that $\|x^n(a)\| = \langle 1_K, x^n(a)\rangle$ and

$$\int_A \|x^n(a)\|\, d\mu(a) + \|x^n_\infty\| = \left\| \int_A x^n(a)d\mu(a) + x^n_\infty \right\| = \|\omega^n + \vartheta^n u^n\|\,.$$

Since $\lim_{n\to\infty} \omega^n + \vartheta^n u^n = \omega + u^*$, applying Fatou's lemma, we get a contradiction. □

The sequence $(p^n)_{n\in\mathbb{N}}$ is $\|.\|_\infty$-converges to p^*, it follows that there exists $\eta > 0$, such that for all n large enough, $p^n \geqslant \eta 1_K$.

Claim 4.5.5. The sequence $(x^n)_{n\in\mathbb{N}}$ is integrably bounded and the sequence $(x^n_\infty)_{n\in\mathbb{N}}$ is w^*-convergent to 0.

Proof. We will first prove that $\lim_{n\to\infty} x^n_\infty = 0$. For all $n \in \mathbb{N}$, p^n lie in a $\|.\|_\infty$-compact set K. Without any loss of generality we can suppose that for all $n \in \mathbb{N}$, $\|p^n\| \leqslant 1$ and $p^n \geqslant \eta 1_K$. From (4.2), for all $n \in \mathbb{N}$,

$$\eta \|x^n_\infty\| \leqslant \sum_{j\in J} (\vartheta^n - \vartheta^n_j)| \langle p^n, y^n_j \rangle |.$$

Since for each $j \in J$, $\lim\limits_{n\to\infty} \langle p^n, y^n_i\rangle = \langle p^*, y^*_j\rangle$, it follows that $\lim\limits_{n\to\infty} \|x^n_\infty = 0\|$.

We prove now that the sequence $(x^n)_{n\in\mathbb{N}}$ is integrably bounded. Let $A' \in \mathcal{A}$ be a measurable subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost everwhere* assumptions and properties are satisfied for all $a \in A'$. Let $a \in A'$, from (4.1), for all $n \in \mathbb{N}$,

$$\langle p^n, x^n(a)\rangle = \langle p^n, e^n(a)\rangle + \sum_{j\in J} \theta^n_j(a) \langle p^n, y^n_j\rangle\,.$$

Since for all $j \in J$, $\lim_{n\to\infty} \langle p^n, y^n_j\rangle = \langle p^*, y^*_j\rangle$, there exists $M > 0$ such that

$$\eta \|x^n(a)\| \leqslant \|e^n(a)\| + M \sum_{j\in J} \theta^n_j(a).$$

Following Fact (4.5.1), for all $n \in \mathbb{N}$,

$$\|x^n(a)\| \leqslant \frac{(1 + M)(1 + g(a))}{\eta}.$$

□

Applying Theorem 4.7.1 and passing to a subsequence if necessary, there exists a Gelfand integrable function $x^* : A \to M(T)$, such that

$$\int_A x^*(a)d\mu(a) = w^* - \lim_{n\to\infty} \int_A x^n(a)d\mu(a),$$

$$\text{for a.e. } a \in A^{na} \quad x^*(a) \in w^* - \overline{\text{co}}\, [w^* - \text{ls}\,\{x^n(a)\}]$$

and

$$\text{for all } a \in A^{pa} \quad x^*(a) \in w^* - \text{ls}\,\{x^n(a)\},$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$.

4.5.3 The element (x^*, y^*, p^*) is a quasi-equilibrium of $\mathcal{E}$

The condition (b) of the definition of a quasi-equilibrium has already been proved in Claim 4.5.2.

Since $\lim_{n\to\infty} \int_A x^n(a)d\mu(a) = \omega + \sum_{j\in J} y_j^*$, to get the condition (c) of the definition of a quasi-equilibrium for the economy $\mathcal{E}$, it is sufficient to prove that $x^* \in S^1(X)$.
We recall that

$$A_0 = \bigcup_{n\in\mathbb{N}} A \setminus (\cup_{i\in I^n} A_i^n).$$

Let A' be a subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost everwhere* assumptions and properties are satisfied for all $a \in A'$. We propose to prove that, for all $a \in A'$, $x^*(a) \in X(a)$. Let $a \in A'$, by construction, we have that for every $n \in \mathbb{N}$, $x^n(a) \in X^n(a)$, and thus, for every $n \in \mathbb{N}$, $d(x^n(a), X^n(a)) = 0$. Let $m \in d\text{-ls}\{x^n(a)\}$, applying Fact 4.5.1, $d(m, X(a)) = 0$. Since $X(a)$ is d-closed, it means that $m \in X(a)$. Thus $d\text{-ls}\{x^n(a)\} \subset X(a)$, and under Assumption C, it follows that $x^*(a) \in X(a)$.

We will now prove that (x^*, y^*, p^*) satisfies the condition (a') of the definition of a quasi-equilibrium of $\mathcal{E}$. Let $a \in A'$. First, with (4.1), Proposition 4.5.2 and Fact 4.5.1, we easily check that

$$\langle p^*, x^*(a)\rangle = \langle p^*, e(a)\rangle + \sum_{j\in J} \theta_j(a) \left\langle p^*, y_j^*\right\rangle.$$

Second, we will prove that

$$\forall x' \in P_a(x^*(a)) \quad \langle p^*, x'\rangle \geqslant \langle p^*, x^*(a)\rangle.$$

Let $x' \in P_a(x^*(a))$. Since $X(a) = d\text{-cl}\{f_k(a) \mid k \in \mathbb{N}\}$, we can suppose (extracting a subsequence if necessary) that $(f_k(a))_{k\in\mathbb{N}}$ is d-convergent to x'. But $P_a(x^*(a))$ is d-open in $X(a)$, thus there exists $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$, $f_k(a) \in P_a(x^*(a))$. To prove that $\langle p^*, x'\rangle \geqslant \langle p^*, x^*(a)\rangle$, it is sufficient to prove that for all k large enough, $\langle p^*, f_k(a)\rangle \geqslant \langle p^*, x^*(a)\rangle$.
Now, let $k \geqslant k_0$.

Claim 4.5.6. There exist an increasing application $\varphi : \mathbb{N} \to \mathbb{N}$ and such that:

$$\forall n \in \mathbb{N} \quad f_k(a) \in P^{\varphi(n)}\left(a, x^{\varphi(n)}(a)\right).$$

Proof. Suppose that for all increasing application $\varphi : \mathbb{N} \to \mathbb{N}$, there exists an increasing application $\phi : \mathbb{N} \to \mathbb{N}$, such that:

$$\forall n \in \mathbb{N} \quad d\left(x^{\varphi\circ\phi(n)}(a), R_k^{\varphi\circ\phi(n)}(a)\right) = 0.$$

Applying Fact 4.5.1, it follows that for all $\ell \in d\text{-ls}\{x^n(a) \mid n \in \mathbb{N}\}$, $d(\ell, R_k(a)) = 0$. Then following Assumption C, $d\text{-}\overline{\text{co}}\,[d\text{-ls}\{x^n(a) \mid n \in \mathbb{N}\}] \subset R_k(a)$, if a belongs to the non-atomic part of $(A, \mathcal{A}, \mu)$, and $d\text{-ls}\{x^n(a) \mid n \in \mathbb{N}\} \subset R_k(a)$ elsewhere. It follows that $x^*(a) \in R_k(a)$, i.e. , $f_k(a) \notin P_a(x^*(a))$. Contradiction. □

With claim 4.5.6 and (4.1), for all $n \in \mathbb{N}$,

$$\left\langle p^{\varphi(n)}, f_k^{\varphi(n)}(a)\right\rangle \geqslant \left\langle p^{\varphi(n)}, e^{\varphi(n)}(a)\right\rangle + \sum_{j\in J} \theta_j^{\varphi(n)}(a) \left\langle p^{\varphi(n)}, y_j^{\varphi(n)}\right\rangle.$$

Passing to the limit, we get that

$$\langle p^*, f_k(a)\rangle \geqslant \langle p^*, e(a)\rangle + \sum_{j\in J} \theta_j(a) \left\langle p^*, y_j^*\right\rangle = \langle p^*, x^*(a)\rangle.$$

4.6 Appendix A : Large economies with finitely many agents

4.6.1 The Model and the equilibrium concepts

We consider a production economy with a commodity space $\mathbb{L}$ and a price space $\mathbb{P}$, which are both linear vector spaces such that $\langle \mathbb{P}, \mathbb{L}\rangle$ is a dual pair [16]. Let I be the finite set of agents (or consumers). An agent $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{L}$, an initial endowment $e_i \in \mathbb{L}$ and a preference relation described by a correspondence P_i from $\prod_{i\in I} X_i$ into X_i. A consumption plan x is an element of $\prod_{i\in I} X_i$ and a consumption bundle x_i of agent $i \in I$ is an element of X_i. The aggregate consumption set is noted $X_\Sigma := \sum_{i\in I} X_i$ and the aggregate initial endowment is noted $\omega := \sum_{i\in I} e_i$. Consider a consumption plan $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$, for an agent $i \in I$, the set $P_i(x) \subset X_i$ is the set of consumption bundles strictly preferred to x_i by the i-th agent, given the consumption bundles $(x_k)_{k\neq i}$ of the other consumers. Let J be the finite set of firms (or producers). A firm $j \in J$ is characterized by a production set $Y_j \subset \mathbb{L}$. The set of production plans is $\prod_{j\in J} Y_j$ and the aggregate production set is $Y_\Sigma := \sum_{j\in J} Y_j$. The profit made by the firm $j \in J$ is distributed among the consumers following a share function $\theta_j := (\theta_{ij})_{i\in I}$, such that for all $i \in I$, $\theta_{ij} \geqslant 0$ and $\sum_{i\in I} \theta_{ij} = 1$.

A complete description of a production economy $\mathcal{E}$ is given by the following list:

$$\mathcal{E} := \left(\langle \mathbb{P}, \mathbb{L}\rangle, (X_i, P_i, e_i)_{i\in I}, (Y_j, \theta_j)_{j\in J}\right).$$

Definition 4.6.1. An element (x^*, y^*, p^*) of $\prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathbb{P}$ is a *Walrasian equilibrium* of the production economy $\mathcal{E}$ if $p^* \neq 0$ and the following conditions are satisfied.

(a) For every $i \in I$,

$$\langle p^*, x_i^*\rangle = \langle p^*, e_i\rangle + \sum_{j\in J} \theta_{ij} \left\langle p^*, y_j^*\right\rangle \quad \text{and} \quad x \in P_i(x^*) \Longrightarrow \langle p^*, x\rangle > \langle p^*, x_i^*\rangle .$$

(b) For every $j \in J$,

$$y \in Y_j \Longrightarrow \langle p^*, y\rangle \leqslant \left\langle p^*, y_j^*\right\rangle .$$

(c)

$$\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^*.$$

An element $(x^*, y^*, p^*) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathbb{P}$ is a *Walrasian quasi-equilibrium* of the production economy $\mathcal{E}$ if $p^* \neq 0$ and if the conditions (b) and (c) together with the following (a') are satisfied.

(a') For every $i \in I$,

$$\langle p^*, x_i^*\rangle = \langle p^*, e_i\rangle + \sum_{j\in J} \theta_{ij} \left\langle p^*, y_j^*\right\rangle \quad \text{and} \quad x \in P_i(x^*) \Longrightarrow \langle p^*, x\rangle \geqslant \langle p^*, x_i^*\rangle .$$

A *non-trivial Walrasian quasi-equilibrium* of a production economy $\mathcal{E}$ is a Walras quasi-equilibrium $(x^*, y^*, p^*) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \mathbb{P}$ satisfying:

(d) there exists an agent $i_0 \in I$ satisfying $\inf \langle p^*, X_{i_0}\rangle < \left\langle p^*, x_{i_0}^*\right\rangle$.

As well-known, under some continuity assumptions on preferences, classical assumptions on production and some irreducibility condition on the economy, a non-trivial Walrasian quasi-equilibrium is easily proved to be a Walrasian equilibrium.

[16] That is $\langle ., .\rangle : \mathbb{P} \times \mathbb{L} \to \mathbb{R}$ is a non-degenerate bilinear form, in the sense that if $\langle p, x\rangle = 0$ for all $p \in \mathbb{P}$, then $x = 0$ and if $\langle p, x\rangle = 0$ for all $x \in \mathbb{L}$, then $p = 0$.

4.6.2 The Assumptions

We now present the list of assumptions that economies will be required to satisfy.

4.6.2.1 Standard Assumptions

Assumption ($\mathbf{T}_f$). *The commodity space $\mathbb{L}$ is endowed with two Hausdorff linear topologies τ and σ such that $(\mathbb{L}, \tau)$ is locally convex and $\mathbb{P}$ is the topological dual of $(\mathbb{L}, \tau)$. The duality $\langle .,. \rangle$ coincide with the natural evaluation, that is for all $(p, x) \in \mathbb{P} \times \mathbb{L}$, $\langle p, x \rangle = p(x)$.*

Assumption ($\mathbf{C}_f$). *For each consumer $i \in I$, X_i is convex σ-closed, for each $z_i \in X_i$, $P_i^{-1}(z_i)$ [17] is σ^I-open in $\prod_{i \in I} X_i$ and for each consumption plan $x = (x_i) \in \prod_{i \in I} X_i$, $x_i \notin \operatorname{co} P_i(x)$.*

Assumption ($\mathbf{P}_f$). *The aggregate production set Y_Σ is convex and σ-closed.*

Assumption ($\mathbf{B}_f$). *The set of realizable consumption plans $\mathcal{A}_X(\mathcal{E})$ is σ^I-compact in $\prod_{i \in I} X_i$, where*

$$\mathcal{A}_X(\mathcal{E}) := \left\{ x \in \prod_{i \in I} X_i \;\middle|\; \sum_{i \in I} x_i \in \{\omega\} + Y_\Sigma \right\}.$$

Assumption ($\mathbf{S}_f$). *For all $i \in I$,*

$$e_i \in X_i - \sum_{j \in J} \theta_{ij} \overline{\operatorname{co}}\, Y_j.$$

4.6.2.2 Lattice Assumptions

Assumption ($\mathbf{L}_f$). *The vector space $\mathbb{L}$ is endowed with a partial linear order $\geqslant$ such that $(\mathbb{L}, \geqslant)$ is a linear vector lattice [18] (or a Riesz space) with a τ-closed positive cone $\mathbb{L}_+ = \{x \in \mathbb{L} \mid x \geqslant 0\}$. The dual space $\mathbb{P}$ endowed the dual order is a sublattice of the order dual [19].*

Assumption ($\mathbf{K}_f$). *Order intervals $[x, y] = \{z \in \mathbb{L} \mid x \leqslant z \leqslant y\}$ in $\mathbb{L}$ are σ-compact.*

Assumption ($\mathbf{SP}_f$). *For all open symmetric τ-neighborhood V of zero in $\mathbb{L}$, the set $V^\circ \vee V^\circ$ is relatively $\sigma(\mathbb{P}, \mathbb{L})$-compact, where V° is the polar [20] set of V and*

$$V^\circ \vee V^\circ = \{\pi \in \mathbb{P} \mid \exists (p, q) \in V^\circ \times V^\circ \quad \pi = p \vee q\}.$$

Remark 4.6.1. Assumption SP_f is not classical in the literature of economies with a vector lattice commodity space. Note first that if the topology τ is locally solid [21] then Assumption SP_f is automatically [22] valid. Moreover if the topologies τ and σ coincide, then Assumption SP_f is a consequence of Assumptions L_f and K_f (Proposition 4.7.9). In particular, for economies with differentiated commodities, if τ and σ coincide with the bounded weak star topology bw^* then (Proposition 4.7.10) the following set $K(V)$ is $\|.\|_\infty$-relatively compact,

$$K(V) = \{p_1 \vee \cdots \vee p_n \in C(T) \mid n \geqslant 1 \quad \text{and} \quad \forall i \in \{1, \cdots, n\} \quad p_i \in V^\circ\}.$$

Moreover if $(M, \mathcal{M}, \nu)$ is a measure space, let us condiser, for $1 \leqslant p < \infty$, the Lebesgue spaces $L^p(M, \mathcal{M}, \nu)$. For the price-commodity pairing $\langle L^q, L^p \rangle$, endowed with $\tau = \|.\|_p$, Assumption SP_f is satisfied.

[17] We let $P_i^{-1}(z_i) = \{x \in \prod_{i \in I} X_i \mid z_i \in P_i(x)\}$.

[18] An ordered vector space $(\mathbb{L}, \geqslant)$ is a vector lattice if for all $x, y \in \mathbb{L}$, the least upper bound, noted $x \vee y$, exists in $\mathbb{L}$ and the least lower bound, noted $x \wedge y$, exists in $\mathbb{L}$.

[19] We refer to Aliprantis and Burkinshaw [6] for precisions

[20] That $V^\circ = \{p \in \mathbb{P} \mid \forall v \in V \quad |\langle p, v \rangle| \leqslant 1\}$.

[21] That is τ has a base at zero consisting of solid neighborhoods. A subset X of $\mathbb{L}$ is solid if $|x| \leqslant |y|$ and $y \in X$ imply $x \in X$. We refer to [6] for precisions.

[22] If $V \subset \mathbb{L}$ is a solid set then $V^\circ \subset \mathbb{P}$ is a solid set and $V^\circ \vee V^\circ \subset 4V^\circ$ is relatively $\sigma(\mathbb{P}, \mathbb{L})$-compact.

4.6.2.3 Properness Assumptions

Assumption (S'$_f$). *There exists a set $A \subset \mathbb{L}$ radial*[23] *at 0 such that either there exists $i \in I$ such that $X_i + A \cap \mathbb{L}_+ \subset X_i$ or there exists $j \in J$ such that $\overline{\mathrm{co}}\, Y_j - A \cap \mathbb{L}_+ \subset \overline{\mathrm{co}}\, Y_j$.*

Assumption (I$_f$). *There exists $V \subset \mathbb{L}$ a τ-neighborhood of 0 such that one of the two following properties holds.*

(i) *There exists $j \in J$ and $y_j \in Y_j$ such that $\{y_j\} + V \cap \mathbb{L}_+ \subset \overline{\mathrm{co}}\, Y_j$.*

(ii) *There exists $i \in I$ and $x \in \prod_{i \in I} X_i$ such that $\{x_i\} + V \cap \mathbb{L}_+ \subset X_i$ and $P_i(x)$ has a τ-interior point in X_i.*

Assumption (UP$_f$). *There exists a τ-open cone Γ, such that $\Gamma \cap \mathbb{L}_+ \neq \emptyset$ and there exists a set $A \subset \mathbb{L}$ radial at 0 such that for all $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ with $\sum_{i \in I} x_i - \sum_{j \in J} y_j - \omega \in A \cap \mathbb{L}_+$, the following properties are satisfied.*

(a) *For every $i \in I$ there exists a set $A^i_{x_i} \subset \mathbb{L}$ radial at x_i, such that*

$$(\{x_i\} + \Gamma) \cap \{z \in \mathbb{L} \mid z \geqslant x_i \wedge e_i\} \cap A^i_{x_i} \subset \overline{\mathrm{co}}\, P_i(x).$$

(b) *For every $j \in J$ there exists a set $A^j_{y_j} \subset \mathbb{L}$ radial at y_j, such that*

$$(\{y_j\} - \Gamma) \cap \{z \in \mathbb{L} \mid z \leqslant y_j \vee 0\} \cap A^j_{y_j} \subset \overline{\mathrm{co}}\, Y_j.$$

Remark 4.6.2. This assumption is borrowed from the F-properness assumption introduced by Podczeck [32] for pure exchange economies with finitely many agents and adapted to production economies by Florenzano and Marakulin [21].

Remark 4.6.3. In Assumption UP$_f$, property (a) is the asymmetric part of the uniform properness for exchange economies developed in Mas-Colell [30] and property (b) is the asymmetric part of the uniform properness developed for production economies in Richard [36].

Remark 4.6.4. Following the proof of the existence theorem, we can replace the condition (b) by the following condition (b').

(b') There exists a subset A' of $M(T)$, radial at $u = \sum_{j \in J} y_j$, such that

$$(\{u\} - \Gamma) \cap \{z \in \mathbb{L} \mid z \leqslant u \vee 0\} \cap A' \subset Y_\Sigma.$$

4.6.3 Existence Result

Theorem 4.6.1. *Let $\mathcal{E}$ be an economy with finitely many consumers satisfying Standard, Lattice and Properness Assumptions.*

(a) *If $v \in \Gamma \cap \mathbb{L}_+$ is a properness vector, then the economy $\mathcal{E}$ has a quasi-equilibrium (x^*, y^*, p^*) such that $\langle p^*, v \rangle = 1$.*

(b) *If moreover $v \in \{\omega\} + Y_\Sigma - X_\Sigma$, then (x^*, y^*, p^*) is a non-trivial quasi-equilibrium of $\mathcal{E}$.*

4.6.4 Proof of Theorem 4.6.1

Let $\mathcal{E}$ be an economy with finitely many consumers satisfying Standard, Lattice and Properness Assumptions and let $v \in \Gamma \cap \mathbb{L}_+$ be a properness vector. Note that $v > 0$ since Γ is open.

[23] We recall that a subset $R \subset \mathbb{L}$ is radial at $x \in R$ if for all $v \in \mathbb{L}$, there exists $\lambda > 0$ such that the segment $[x, x + \lambda v]$ still lie in R.

4.6.4.1 Approximating economies

Following Zame [44], we construct a net of approximating economies. Since Γ is open, there exists $V \subset \mathbb{L}$ a τ-open convex and symmetric neighborhood of 0, such that $\{v\} + V \subset \Gamma$. Without any loss of generality, we can suppose that Γ is the cone with vertex 0 generated by $\{v\} + V$. Following assumption S_f, there exists for each $i \in I$, $\hat{x}_i \in X_i$ and $(\hat{y}_{ij})_{j\in J} \in \prod_{j\in J} \overline{\mathrm{co}}\, Y_j$ such that $e_i = \hat{x}_i + \sum_{j\in J}\theta_{ij}\hat{y}_{ij}$. Following assumption I_f there exists W a τ-neighborhood of 0 and there exist (to simplify the proof, we suppose that the condition (i) of I_f is satisfied) $j_0 \in J$ and $\tilde{y}_{j_0} \in Y_{j_0}$ such that $\{\tilde{y}_{j_0}\} + W \cap \mathbb{L}_+ \subset \overline{\mathrm{co}}\, Y_{j_0}$. Let $\mathcal{K}$ denote the family of all principal order ideals of $\mathbb{L}$ which contain $\{\tilde{y}_{j_0}\} \cup \{e_i, \hat{x}_i, \hat{y}_{ij} \mid i \in I \text{ and } j \in J\}$. For each $K \in \mathcal{K}$, choose once for all an element $e_K \in \mathbb{L}_+$ which generates K, that is $K = \mathbb{L}(e_K) = \{x \in \mathbb{L} \mid \exists r > 0 \ -re_K \leqslant x \leqslant re_K\}$. For each $x \in K$, define $\|x\|_K = \inf\{r > 0 \mid -re_K \leqslant x \leqslant re_K\}$. The space $(K, \|.\|_K)$ is a normed Riesz space [24].

Fix a principal order ideal $K \in \mathcal{K}$. Fix a positive integer s large enough such that for each $i \in I$, $\{e_i, \hat{x}_i\} \subset \{-se_K\} + K_+$ and for each $j \in J$, $\{\hat{y}_{ij}, \tilde{y}_{j_0}\} \subset \{+se_K\} - K_+$. Let $t > 0$, we consider $\mathcal{E}(K, s, t)$ the economy with the set I of consumers and the set J of firms, defined as follows. For each $i \in I$ the consumptions set is $X_i(K, s, t) := X_i \cap (\{-se_K\} + K_+)$, for each consumption plan $x \in \prod_{i\in I} X_i(K, s, t)$ the strict preferred set is $P_i(K, s, t)(x) = [\mathrm{co}\, P_i(x)] \cap X_i(K, s, t)$. Following Assumption $\mathrm{S'}_f$, there exits a set $A^S \subset \mathbb{L}$ radial at 0 and $i_S \in I$ such that either $X_{i_S} + A^S \cap \mathbb{L}_+ \subset X_{i_S}$ or $\sum_{j\in J}\theta_{i_S j}\overline{\mathrm{co}}\, Y_j - A^S \cap \mathbb{L}_+ \subset \sum_{j\in J}\theta_{i_S j}\overline{\mathrm{co}}\, Y_j$. Then for each $i \neq i_S$ the initial endowment is $e_i(K, s, t) = e_i$ and $e_{i_S}(K, s, t) = e_{i_S} + (1/t)e_K$. We let $\omega(K, s, t)$ denote the aggregate initial endowment. Note that $\omega(K, s, t) = \omega + (1/t)e_K$. For each producer $j \in J$ the production set is $Y_j(K, s, t) = [\overline{\mathrm{co}}\, Y_j] \cap (\{+se_K\} - K_+)$ and the shares are not changed, for all $i \in I$, $\theta_{ij}(K, s, t) = \theta_{ij}$.

Since $\mathcal{E}$ satisfies Assumption $\mathrm{S'}_f$, for all $t > 0$ be large enough, $\mathcal{E}(K, s, t)$ satisfies Assumption S_f. As noticed by Podczeck in [32], under Assumptions L_f the norm topology $\|.\|_K$ is finer than τ_K the topology induced by τ on K. Now we assert that for all t large enough, the economy $\mathcal{E}(K, s, t)$ has an Edgeworth equilibrium. Indeed the economy $\mathcal{E}(K, s, t)$ satisfies all the assumptions [25] of Proposition 3 in Florenzano [20]. Let $x(K, s, t) \in \prod_{i\in I} X_i(K, s, t)$ be an Edgeworth equilibrium of $\mathcal{E}(K, s, t)$, then $0 \notin G$ where G is the $\mathbb{Q}$-convex hull [26] of the following set

$$\bigcup_{i\in I}\left(P_i(K, s, t)(x(K, s, t)) - \{e_i(K, s, t)\} - \sum_{j\in J}\theta_{ij}Y_j(K, s, t)\right).$$

Then, applying Assumption [27] I_f, the set G has a $\|.\|_K$-interior point. Applying Proposition 4.7.8 there exists a non-zero price $p(K, s, t) \in (K, \|.\|_K)'$ separating 0 and G. Moreover, since preferences satisfy Assumptions UP_f, for each $i \in I$ the strict preferred set $P_i(x(K, s, t))$ is $\|.\|_K$-locally non-satiated. If we let $y(K, s, t) \in \prod_{j\in J} Y_j(K, s, t)$ be such that $\sum_{i\in I}x_i(K, s, t) = \sum_{i\in I}e_i(X, s, t) + \sum_{j\in J}y_j(K, s, t)$, then $(x(K, s, t), y(K, s, t), p(K, s, t))$ is a Walrasian quasi-equilibrium of $\mathcal{E}(K, s, t)$.

4.6.4.2 Price equilibria and Properness assumption

It is straightforward to verify that we can apply the first part of Proposition 2.1 in Florenzano and Marakulin [21] to the economy $\mathcal{E}(K, s, t)$ in order to obtain the following claim.

Claim 4.6.1. There exists, for all $k \in I \cup J$, τ-continuous linear functionals $\pi_k(K, s, t) \in \mathbb{P}$ such that $\pi_k(K, s, t) \leqslant p(K, s, t)$ and

$$\forall k \in I \cup J \quad \langle \pi_k(K, s, t), \Gamma\rangle \geqslant 0. \tag{4.7}$$

Moreover, if we let $\pi(K, s, t) = \bigvee_{k\in I\cup J} \pi_k(K, s, t)$, then

$$\pi(K, s, t) \in \mathbb{P} \quad \text{and} \quad \pi(K, s, t)_{|K} \leqslant p(K, s, t)\,, \tag{4.8}$$

[24] That is $|x| \leqslant |y|$ in K implies $\|x\|_K \leqslant \|y\|_K$.

[25] In fact the only assumption whose verification is not trivial is the σ^I_K-compactness of the realizable consumption plans $\mathcal{A}_X(\mathcal{E}(K, s, t))$. It is a consequence of Assumptions K.

[26] We refer to Proposition 4.7.8 for the definition of the $\mathbb{Q}$-convex hull.

[27] This assumption is automatically valid in Podczeck [32] since consumption sets are comprehensive, but surprisingly this assumption does not appear in Florenzano and Marakulin [21].

and for all $z \in \sum_{i \in I} Z_i - \sum_{j \in J} Z_j$ such that $z \leqslant \omega(K,s,t)$,

$$\langle \pi(K,s,t), \omega(K,s,t) - z \rangle = \langle p(K,s,t), \omega(K,s,t) - z \rangle\,, \tag{4.9}$$

where for all $i \in I$ and $j \in J$ we let $Z_i = \{z \in \mathbb{L} \mid z \geqslant x_i(K,s,t) \wedge e_i\}$ and $Z_j = \{z \in \mathbb{L} \mid z \leqslant y_j(K,s,t) \vee 0\}$.

Following (4.8), $\pi(K,s,t)_{|K} - p(K,s,t)$ is non positive on K_+. Applying Assumption UP_f and (4.9), it vanishes on $\omega(K,s,t) - \omega$. Note that $\omega(K,s,t) - \omega$ lies in the $\|.\|_K$-interior of K_+. It follows that $\pi(K,s,t)_{|K} = p(K,s,t)$. Following (4.7) and since $v > 0$,

$$\forall k \in I \cup J \quad \langle \pi_k(K,s,t), V \rangle \leqslant \langle \pi_k(K,s,t), v \rangle \leqslant \langle \pi(K,s,t), v \rangle\,.$$

Hence, if $\langle \pi(K,s,t), v \rangle = 0$ then $\pi(K,s,t) = 0$. But $\pi(K,s,t)$ coincide on K with the non-zero functional $p(K,s,t)$. Contradiction. We can thus suppose that

$$\langle \pi(K,s,t), v \rangle = 1 \quad \text{and} \quad \forall k \in I \cup J \quad \langle \pi_k(K,s,t), V \rangle \leqslant 1.$$

Applying Assumption SP_f, for all $K \in \mathcal{K}$, for all s large enough and for all t large enough, $\pi(K,s,t)$ lies in a $\sigma(\mathbb{P},\mathbb{L})$-compact set $G(V)$.

4.6.4.3 Convergence when $t \to \infty$

We fix two of the three parameters: a principal order ideal $K \in \mathcal{K}$ and a positive integer s large enough. For each positive integer t large enough, the consumption plan $x(K,s,t)$ is realizable. We then check that there exists [28] $M > 0$ such that

$$\forall i \in I \quad x_i(K,s,t) \in [-se_K, \omega + Me_K]\,.$$

Following Structural Assumptions, for all t large enough, $x(K,s,t)$ lie in a σ^I-compact set. Moreover, consumption sets are σ-closed, we can thus suppose (passing to a subsequence if necessary) that the sequence $(x(K,s,t))_{t \geqslant 1}$ is σ^I-convergent to a consumption plan $x(K,s) \in \prod_{i \in I} X_i$. Since for for all $t > 0$, $\omega(K,s,t) = \omega + (1/t)e_K$, the sequence $(\omega(K,s,t))_{t \geqslant 1}$ is σ convergent to ω. The aggregate production set Y_Σ is σ-closed, it follows that $x(K,s)$ is realizable, that is $x(K,s) \in \mathcal{A}_X(\mathcal{E})$.

Following Assumption SP_f and passing to a subsequence if necessary, $(\pi(K,s,t))_{t \geqslant 1}$ is $\sigma(\mathbb{P},\mathbb{L})$-convergent to a price $\pi(K,s) \in G(V)$ satisfying $\langle \pi(K,s), v \rangle = 1$.

4.6.4.4 Convergence when $s \to \infty$

We fix a principal ideal $K \in \mathcal{K}$. We proved that for all integer s large enough, the consumption plan $x(K,s)$ is realizable. Applying Assumption B, we can suppose (extracting a subsequence if necessary) that the sequence $(x(K,s))_{s \geqslant 1}$ is σ^I-convergent to $x(K) \in \prod_{i \in I} X_i$. Once again, since the aggregate production set is σ-closed, the consumption plan $x(K)$ is realizable.

The sequence $(\pi(K,s))_{s \geqslant 1}$ still lie in $G(V)$. Passing to a subsequence if necessary, we can suppose that $(\pi(K,s))_{s \geqslant 1}$ is σ^I-convergent to a price $\pi(K) \in G(V)$ satisfying $\langle \pi(K), v \rangle = 1$.

4.6.4.5 Convergence of the net directed by $\mathcal{K}$

We proved that for all $K \in \mathcal{K}$, the consumption plan $x(K)$ is realizable. Applying Assumption B, we can suppose (extracting a subnet if necessary) that the net $(x(K)_{K \in \mathcal{K}})$ is σ^I-convergent to $x^* \in \prod_{i \in I} X_i$. Once again, since the aggregate production set is σ-closed, the consumption plan x^* is realizable. We let $(y_j^*) \in \prod_{j \in J} Y_j$ be such that $\sum_{i \in I} x_i^* = \omega + \sum_{j \in J} y_j^*$.

The net $(\pi(K)_{K \in \mathcal{K}})$ still lie in $G(V)$. Passing to a subnet if necessary, we can suppose that $(\pi(K))_{K \in \mathcal{K}}$ is σ^I-convergent to a price $\pi^* \in G(V)$ satisfying $\langle \pi^*, v \rangle = 1$.

[28] Take $M = 1/t + \mathrm{Card} J + \mathrm{Card} I - 1$.

4.6.4.6 Existence of a quasi-equilibrium

We prove now that (x^*, y^*, π^*) is a quasi-equilibrium of the economy $\mathcal{E}$. In particular, as $\langle \pi^*, v \rangle = 1$, the price π^* is not zero. Let $i \in I$ and $(x_i, y) \in X_i \times \prod_{j \in J} Y_j$. Under Assumption C_f, there exists $K_0 \in \mathcal{K}$ such that for all $K \in \mathcal{K}$ containing K_0, $x_i \in P_i(x(K))$ and $(x, y) \in K \times K^J$. Let $K \in \mathcal{K}$ such that $K_0 \subset K$. Under Assumption C_f, there exists an integer $s(K)$ such that for all $s > s(K)$, $x_i \in P_i(x(K, s))$ and for each $j \in J$, $y_j \leqslant se_K$. Let $s > s(K)$, under Assumption C_f, there exists an integer $t(K, s)$ such that for all $t > t(K, s)$, $x_i \in P_i(x(K, s, t))$ and $y \in \prod_{j \in J} Y_j(K, s, t)$. Since $(x(K, s, t), y(K, s, t), \pi(K, s, t)_{|K})$ is a quasi-equilibrium of $\mathcal{E}(K, s, t)$, it follows that

$$\langle \pi(K, s, t), x_i \rangle \geqslant \langle \pi(K, s, t), e_i(K, s, t) \rangle + \sum_{j \in J} \theta_{ij} \langle \pi(K, s, t), y_j \rangle .$$

Following a simple limit argument, $\langle \pi^*, x_i \rangle \geqslant \langle \pi^*, e_i \rangle + \sum_{j \in J} \theta_{ij} \langle \pi^*, y_j \rangle$. Under Assumption UP_f the preferences are τ-locally non-satiated. We check that (x^*, y^*, π^*) is a quasi-equilibrium of $\mathcal{E}$.

4.7 Appendix B : Mathematical auxiliary results

4.7.1 Asymptotic cones

Following Section 4.2, we recall that if X is a subset of $M(T)$, then we let $A_{w^*}(X)$ be the set of elements $x \in L$ such that $x = w^*\text{-}\lim_{n \to \infty} \lambda_n x_n$ where $(\lambda_n)_{n \in \mathbb{N}}$ is a real sequence decreasing to 0 and $(x_n)_{n \in \mathbb{N}}$ is a sequence in X.

Proposition 4.7.1. *Let X, Y two subsets of $M(T)$, with $X \subset M(T)_+$. If $A_{w^*}(X) \cap A_{w^*}(Y) = \{0\}$, then $X \cap Y$ is $\|.\|$-bounded.*

Proof. Suppose in the contrary, that $X \cap Y$ is not $\|.\|$-bounded. We can thus extract a sequence $(x^n)_{n \in \mathbb{N}}$ in $X \cap Y$, such that for all $n \in \mathbb{N}$, $\|x^n\| \geqslant n$. Let, for all $n \in \mathbb{N}$, $v^n := \frac{x^n}{\|x^n\|}$. By the Banach-Alaoglu Theorem, we can suppose, without any loss of generality, that the sequence $(v^n)_{n \in \mathbb{N}}$ is w^*-convergent to $v \in M(T)$. Since for all $n \in \mathbb{N}$, $v^n \geqslant 0$, then $\langle 1_K, v^n \rangle = \|v^n\|$. Passing to the limit, we get that $\langle 1_K, v \rangle = 1$ and then $v \neq 0$. But $v \in A_{w^*}(X) \cap A_{w^*}(Y)$. Contradiction. $\square$

4.7.2 Measurability of correspondences

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a complete separable metric space. A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *measurable* if for all open set $G \subset D$, $F^-(G) = \{a \in A \mid F(a) \cap G \neq \emptyset\} \in \mathcal{A}$. The correspondence F is said to be *graph measurable* if $\{(a, x) \in A \times D \mid x \in F(a)\} \in \mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. The set of measurable selections of F is noted $S(F)$.

Following Castaing and Valadier [13] and Himmelberg [23], we recall the two following classical characterizations of measurable correspondences.

Proposition 4.7.2. *Consider $F : A \twoheadrightarrow D$ a correspondence with non-empty closed values. The following properties are equivalent.*

(i) *The correspondence F is measurable.*

(ii) *There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable selections of F such that for all $a \in A$, $F(a) = \mathrm{cl}\,\{f_n(a) \mid n \in \mathbb{N}\}$.*

(iii) *For each $x \in D$, the function $\delta_F(., x) : a \mapsto d(x, F(a))$ is measurable.*

Proposition 4.7.3. *Consider $F : A \twoheadrightarrow D$ a correspondence.*

(i) *If F has non-empty closed values then the measurability of F implies the graph measurability of F.*

(ii) *If $(A, \mathcal{A}, \mu)$ is complete then the graph measurability of F implies the measurability of F.*

(iii) *If F has non-empty closed values and $(A, \mathcal{A}, \mu)$ is complete then measurability and graph measurability of F are equivalent.*

Following Aumann [10], graph measurable correspondences (possibly without closed values) have measurable selections.

Proposition 4.7.4. *Consider F a graph measurable correspondence from A into D with non-empty values. If $(A, \mathcal{A}, \mu)$ is complete then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of measurable selections of F, such that for all $a \in A$, $(z_n(a))_{n\in\mathbb{N}}$ is dense in $F(a)$.*

4.7.3 Measurability of preference relations

We consider $(A, \mathcal{A}, \mu)$ a measure space and (D, d) a complete separable metric space. Let P be a correspondence from A into $D \times D$. For each function $x : A \to D$ the *upper section relative to x* is noted $P_x : A \twoheadrightarrow D$ and is defined by $a \mapsto \{y \in D \mid (x(a), y) \in P(a)\}$. For each function $y : A \to D$ the *lower section relative to y* is noted $P^y : A \twoheadrightarrow D$ and is defined by $a \mapsto \{x \in D \mid (x, y(a)) \in P(a)\}$.

Let $X : A \twoheadrightarrow D$ be a correspondence. A *correspondence of preference relations in X* is a correspondence P from A into $D \times D$ satisfying for all $a \in A$, $P(a) \subset X(a) \times X(a)$. For each $a \in A$, we note P_a the correspondence[29] from $X(a)$ into $X(a)$ defined by $x \mapsto \{y \in X(a) \mid (x, y) \in P(a)\}$. For each $y \in X(a)$ the lower inverse image of y by P_a is noted $P_a^{-1}(y) = \{x \in X(a) \mid y \in P_a(x)\}$. The correspondence of preference relations P in X is graph measurable if

$$\{(a, x, y) \in A \times D \times D \mid (x, y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D) \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *Aumann measurable* if

$$\forall (x, y) \in S(X) \times S(X) \quad \{a \in A \mid (x(a), y(a)) \in P(a)\} \in \mathcal{A}.$$

The correspondence of preference relations P in X is *lower graph measurable* if for all measurable selection y of X, the correspondence P^y is graph measurable, that is

$$\forall y \in S(X) \quad G_{P^y} = \{(a, x) \in A \times D \mid (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

The correspondence of preference relations P in X is *upper graph measurable* if for all measurable selection x of X, the correspondence P_x is graph measurable, that is

$$\forall x \in S(X) \quad G_{P_x} = \{(a, y) \in A \times D \mid (x(a), y) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(D).$$

We propose to compare these three concepts of measurability of preference relations.

Proposition 4.7.5. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Then graph measurability of P implies lower and upper graph measurability of P, and lower or upper graph measurability of P implies Aumann measurability of P.*

Proof. This is a direct consequence of Projection Theorem in Castaing and Valadier [13]. □

Under additional assumptions, the converse is true.

Proposition 4.7.6. *Let P be a correspondence of preference relations in X. We suppose that $(A, \mathcal{A}, \mu)$ is complete and that X has a measurable graph. Moreover, we suppose that for a.e. $a \in A$, $X(a)$ is a closed connected subset of D, $P(a)$ is an ordered binary relation on $X(a)$ and for each $x \in X(a)$, $P_a(x)$ and $P_a^{-1}(x)$ are open in $X(a)$. Then Aumann measurability of P implies lower and upper graph measurability of P, and lower and upper graph measurability of P implies graph measurability of P.*

[29] Remark that the graph of P_a and $P(a)$ coincide.

The proof of Proposition 4.7.6 is given in Martins Da Rocha [28]. A direct corollary of Proposition 4.7.3 is the following result.

Proposition 4.7.7. *If for all $a \in A$, for all $y \in X(a)$, $P^{-1}(a, y)$ is d-open in $X(a)$, then P is lower graph measurable if and only if for all measurable selection $x \in S(X)$ the correspondence R_x is measurable.*

4.7.4 Compactness and integrable functions

In this subsection, $(A, \mathcal{A}, \mu)$ is supposed to be a finite and complete measure space.

Theorem 4.7.1. *Let $(f^n)_{n\in\mathbb{N}}$ a sequence of Gelfand integrable functions from A into $M(T)$. If $(f^n)_{n\in\mathbb{N}}$ is integrably bounded, then there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a Gelfand integrable function f^* from A to $M(T)$, such that*

$$w^*\text{-}\lim_{n\to\infty} \int_A f^{\phi(n)}(a) d\mu(a) \;=\; \int_A f^*(a) d\mu(a)\,,$$

$$\textit{for a.e. } \; a \in A^{na} \quad f^*(a) \in w^*\text{-}\overline{\text{co}}\left[w^*\text{-}ls\{f^{\phi(n)}(a)\}\right]$$

and

$$\textit{for all } \; a \in A^{pa} \quad f^*(a) \in w^*\text{-}ls\{f^{\phi(n)}(a)\},$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$.

Proof. Let, for each $n \in \mathbb{N}$, $v^n := \int_A f^n$. Since $(f^n)_{n\in\mathbb{N}}$ is integrably bounded, the sequence $(v^n)_{n\in\mathbb{N}}$ is bounded. It follows that a subsequence of $(v^n)_{n\in\mathbb{N}}$, w^*-converges to some $v^* \in M(T)$. Applying Lemma 6.6 in Podczeck [33] and following the proof of Corollory 4.4 in Balder and Hess [12], the result follows. □

For more precisions about measurability and integration of correspondences, we refer to papers [41] and [42] of Yannelis.

4.7.5 Separation of $\mathbb{Q}$-convex sets

Let $(\mathbb{L}, \tau)$ be a topological vector space. A set G is called $\mathbb{Q}$-convex if for all $x, y \in G$, for all $t \in [0, 1] \cap \mathbb{Q}$, $tx + (1-t)y \in G$. The $\mathbb{Q}$-convex hull of a set G is the smallest $\mathbb{Q}$-convex set containing G. We present hereafter a result of decentralization for a $\mathbb{Q}$-convex set.

Proposition 4.7.8. *Let $(\mathbb{L}, \tau)$ be a topological vector space and G be a $\mathbb{Q}$-convex subset with a τ-interior point and such that $0 \notin G$. Then there exists a non-zero continuous linear functional $p \in (L, \tau)'$ such that*

$$\forall x \in G \quad p(x) \geqslant 0.$$

Proof. The interior $\operatorname{int} G$ of G is a non-empty and $\mathbb{Q}$-convex subset of $\mathbb{L}$. Let $x \in G$, for each $\lambda \in [0, 1[\cap \mathbb{Q}$, $\lambda x + (1-\lambda)u \in \operatorname{int} G$, if $u \in \operatorname{int} G$. It follows that

$$\operatorname{int} G \subset G \subset \operatorname{cl} \operatorname{int} G.$$

Since $\operatorname{int} G$ is τ-open, it is in fact convex. Now $0 \notin \operatorname{int} G$ and we can apply a convex Separation Theorem to provide the existence of a non-zero continuous linear functional $p \in (\mathbb{L}, \tau)'$ such that for all $x \in \operatorname{int} G$, $p(x) \geqslant 0$. With a limit argument, we prove that for all $x \in G$, $p(x) \geqslant 0$. □

4.7.6 Compactness and lattice operations

Proposition 4.7.9. *Let $(\mathbb{L}, \geqslant)$ be a linear vector lattice endowed with an Hausdorff vector topology τ such that the positive cone $\mathbb{L}_+ = \{x \in \mathbb{L} \mid x \geqslant 0\}$ is τ-closed and the dual space $\mathbb{P} = (\mathbb{L}, \tau)'$ endowed the dual order is a sublattice of the order dual. Suppose that order intervals $[x, y] = \{z \in \mathbb{L} \mid x \leqslant z \leqslant y\}$ in $\mathbb{L}$ are τ-compact.*

Then for all open symmetric τ-neighborhood V of zero in $\mathbb{L}$, the set $V^\circ \vee V^\circ$ is relatively $\sigma(\mathbb{P}, \mathbb{L})$-compact, where V° is the polar set [30] of V and

$$V^\circ \vee V^\circ = \left\{p \in \mathbb{P} \mid \exists (p^1, p^2) \in V^\circ \times V^\circ \quad p = p^1 \vee p^2\right\}.$$

Proof. Let $(p_a)_{a\in A}$ be a net[31] of points in $V^\circ \vee V^\circ$. There exists two nets $(p_a^1)_{a\in A}$ and $(p_a^2)_{a\in A}$ of points of V° such that for all $a \in A$, $p_a = p_a^1 \vee p_a^2$. By Alaoglu's Theorem, we can suppose (extracting subnets if necessary) that the nets (p_a^1) and (p_a^2) are $\sigma(\mathbb{P}, \mathbb{L})$-convergent to (respectively) $p_*^1 \in V^\circ$ and $p_*^2 \in V^\circ$. We propose to prove that the net $(p_a)_{a\in A}$ is $\sigma(\mathbb{P}, \mathbb{L})$-convergent to $p_* = p_*^1 \vee p_*^2 \in \mathbb{P}$. Let $x \in \mathbb{L}_+$.

Claim 4.7.1. For each subnet $(p_b)_{b\in B}$ of $(p_a)_{a\in A}$, there exists a subnet $(p_c)_{c\in C}$ such that the net $(p_c(x))_{c\in C}$ is convergent to $p_*(x)$.

Proof. Let $b \in B$ then $p_b(x) = \sup\{p_b^1(x^1) + p_b^2(x^2) \mid (x^1, x^2) \in \Sigma(x)\}$ where $\sigma(x) = \{(x^1, x^2) \in \mathbb{L}_+ \times \mathbb{L}_+ \mid x^1 + x^2 = x\}$. The set $\Sigma(x)$ is τ^2-compact, thus the supremum is attained and there exists two nets $(x_b^1)_{b\in B}$ and $(x_b^2)_{b\in B}$ such that for all $b \in B$, $p_b(x) = p_b^1(x_b^1) + p_b^2(x_b^2)$. Since $\Sigma(x)$ is τ^2-compact, we can suppose (passing to a subnet if necessary) that $(x_b^1)_{b\in B}$ and $(x_b^2)_{b\in B}$ are τ-convergent to $(x_*^1, x_*^2) \in \Sigma(x)$. The evaluation mapping $(q, z) \mapsto q(z)$ restricted to $V^\circ \times \mathbb{L}$ is jointly continuous in the $\sigma(\mathbb{P}, \mathbb{L}) \times \tau$-topology (Theorem 6.46 in Aliprantis and Border [8]). It follows that the sequences $(p_b^1(x_b^1))_{b\in B}$ and $(p_b^2(x_b^2))_{b\in B}$ converge to respectively $p_*^1(x_*^1)$ and $p_*^2(x_*^2)$. And consequently there exists a subnet $(p_c)_{c\in C}$ of $(p_b)_{b\in B}$ such that

$$\lim_{c\in C} p_c(x) = p_*^1(x_*^1) + p_*^2(x_*^2) \leqslant p_*(x).$$

Now $\Sigma(x)$ is τ^2-compact then there exists $(y_*^1, y_*^2) \in \Sigma(x)$ such that

$$p_*(x) = p_*^1(y_*^1) + p_*^2(y_*^2) = \lim_{c\in C}(p_c^1(y_*^1) + p_c^2(y_*^2)).$$

But for each $c \in C$, $p_c^1(y_*^1) + p_c^2(y_*^2) \leqslant p_c(x)$ and passing to the limit, $p_*(x) \leqslant \lim_{c\in C} p_c(x)$. □

Now we are ready to prove that $(p_a(x))_{a\in A}$ converges to $p_*(x)$. Suppose not, then there exist $\varepsilon > 0$ and a subnet $(p_b)_{b\in B}$ such that for all $b \in B$, $|p_b(x) - p_*(x)| > \varepsilon$. Applying Claim 4.7.1 there exists a subnet $(p_c(x))_{c\in C}$ of $(p_b(x))_{b\in B}$ converging to $p_*(x)$. Contradiction.

The space $\mathbb{L}$ is a Riesz space, it particular $\mathbb{L} = \mathbb{L}_+ - \mathbb{L}_+$. It follows that for all $x \in \mathbb{L}$ the net $(p_a(x))_{a\in A}$ converges to $p_*(x)$. This means that the net $(p_a)_{a\in A}$ (in fact a subnet) is $\sigma(\mathbb{P}, \mathbb{L})$-convergent to p_*. □

Proposition 4.7.10. *Let $V \subset M(T)$ be a bw*-neighborhood V of zero. The following set $K(V) \subset C(T)$ is relatively $\|.\|_\infty$-compact,*

$$K(V) = \left\{ \bigvee_{i=1}^{n} p_i \,\middle|\, n \geqslant 1 \quad \text{and} \quad \forall i \in \{1, \cdots, n\} \quad p_i \in V^\circ \right\}.$$

Proof. Note first that the dual order on $C(T)$ coincides with the natural pointwise order on functions, that is, for each $p \in C(T)$, $p \geqslant 0$ if and only if for all $t \in T$, $p(t) \geqslant 0$. Following Holmes [24], without any loss of generality, we can assume that there exists B a $\|.\|_\infty$-compact convex and circled[32] subset of $C(T)$ such that

$$V = \{z \in M(T) \mid \forall p \in B \quad |\langle p, z\rangle| \leqslant 1\}.$$

[30] If V is a subset of $\mathbb{L}$, then the polar set V° (reltive to the duality $\langle \mathbb{P}, \mathbb{L}\rangle$) is $V^\circ := \{p \in \mathbb{P} \mid |\langle p, x\rangle| \leqslant 1 \quad \forall x \in V\}$.
[31] The set of index A is a directed set.
[32] A set A in a vector space X is circled if for each $x \in A$ the line segment joining x and $-x$ lies in A.

We will apply Ascoli's Theorem. We first prove that $K(V)$ is pointwise bounded. Let $t \in T$ then for each $p \in C(T)$, $p(t) = \langle p, \delta_t \rangle$. Now V is a radial subset at 0 in $M(T)$, it follows that there exists $\lambda_t > 0$ such that $\lambda_t \delta_t \in V$ and then for all $p \in V^\circ$, $|p(t)| \leqslant \lambda_t$. We then easily check that for all $p \in K(V)$, $|p(t)| \leqslant \lambda_t$.

We prove now that the set $K(V)$ is equicontinuous. Following the Bipolar Theorem, V° is $\|.\|_\infty$-compact. Let $p \in K(V)$ then there exist an integer $n > 0$ and $p_1, \cdots, p_n \in V^\circ$ such that $p = p_1 \vee \cdots \vee p_n$. It follows that for all $(t, t') \in T^2$, there exists $(i, j) \in \{1, \cdots, n\}$ such that $p(t) - p(t') = q_i(t) - q_j(t')$. By definition of the supremum,

$$q_j(t) - q_j(t') \leqslant p(t) - p(t') \leqslant q_i(t) - q_i(t').$$

It follows that $|p(t) - p(t')| \leqslant \max(|q_j(t) - q_j(t')|, |q_i(t) - q_i(t')|)$. Since V° is $\|.\|_\infty$-compact, it is equicontinuous and the set $K(V)$ is equicontinuous. $\square$

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Résumé

Nous proposons une nouvelle approche pour démontrer l'existence d'équilibres de Walras pour des économies avec un espace mesuré d'agents et un espace des biens de dimension finie ou infinie.

Dans un premier temps (chapitre 1*) on démontre un résultat de discrétisation des correspondances mesurables, qui nous permettra de considérer une économie avec un espace mesuré d'agents comme la limite d'une suite d'économies avec un nombre fini d'agents.*

Dans le cadre des économies avec un espace mesuré d'agents, on applique tout d'abord (chapitre 2*) ce résultat aux économies avec un nombre fini de biens, puis (chapitre* 3*) aux économies avec des biens modélisé par un Banach séparable ordonné par un cône positif d'intérieur non vide, et finalement (chapitre* 4*) aux économies avec des biens différenciés. On parvient ainsi à généraliser les résultats d'existence de Aumann (1966), Schmeidler (1969), Hildenbrand (1970), Khan et Yannelis (1991), Rustichini et Yannelis (1991), Ostroy et Zame (1994) et Podczeck (1997) aux économies avec des préférences non ordonnées et un secteur productif non trivial.*

Mots-clés : *Espace mesuré d'agents, espace des biens de dimension infinie, préférences non ordonnées et discrétisation des correspondances mesurables.*

Abstract

We propose a new approach to prove the existence of Walrasian equilibria for economies with a measure space of agents and a finite or infinite dimensional commodity space.

We begin to prove (in chapter 1*) a discretisation result for measurable correspondences, which allows us to consider an economy with a measure space of agents as the limit of a sequence of economies with a finite, but larger and larger, set of agents.*

In the framework of economies with a measure space of agents, we apply this result, first (in chapter 2*) to economies with finitely many commodities, then (in chapter* 3*) to economies with a separable Banach commodity space ordered by a positive cone which has an interior point, and finally (in chapter* 4*) to economies with differentiated commodities. We generalize existence results of Aumann (1966), Schmeidler (1969), Hildenbrand (1970), Khan and Yannelis (1991), Rustichinni and Yannelis (1991), Ostroy and Zame (1994) and Podczeck (1997) to economies with non ordered preferences and with a non trivial production sector.*

Keywords : *Measure space of agents, possibly infinite dimensional commodity spaces, non ordered preferences and discretization of measurable correspondences.*